THE RATE OF INCREASE OF REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENTIAL-DIFFERENCE EQUATIONS OF THE FIRST ORDER

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1. Introduction. It is the purpose of this paper to prove several theorems describing the rate of increase, as $t \longrightarrow +\infty$, of real solutions of algebraic differential-difference equations of the form

$$(1) \qquad P(t, u(t), u'(t), u(t+1), u'(t+1)) = 0.$$

In this equation, and throughout this paper, $P(t, u, v, \cdots)$ denotes a polynomial in the variables $t, u, v, \cdots$, with real coefficients, and a prime denotes differentiation with respect to t. In order to explain the significance and limitations of these theorems, it is first necessary to summarize the work, by other investigators, which suggested the present discussion.

In 1899, E. Borel, [1], published a memoir in which he studied the magnitude of solutions of algebraic differential equations. His result, as later improved by E. Lindelöf, [4], is quoted here for reference:

Let $u(t)$ be a real function which is defined and which has a continuous first derivative for all t larger than t_0, and which satisfies the first order algebraic differential equation

$$(2) \qquad P(t, u(t), u'(t)) = 0$$

for $t > t_0$. Then there is a positive number k, which depends only on P, such that

$$|u(t)| < \exp(t^k/k)$$

for $t \geq t_0$.

It is noteworthy that it is impossible to prove a result of the above type for higher order equations. For a discussion of this point, refer to Vijayaraghavan, [7].

Received January 10, 1953.
Pacific J. Math. 4 (1954), 483-501