APPROXIMATION OF IMPROPER INTEGRALS BY SUMS OVER MULTIPLES OF IRRATIONAL NUMBERS

R. SHERMAN LEHMAN

1. Introduction and notation. Let α be a positive irrational number. The multiples of α, the numbers $\alpha, 2\alpha, 3\alpha, \cdots$, are equidistributed mod 1. Suppose $f(x)$ is a bounded function, Riemann integrable on the interval $(0,1)$, and periodic with period 1. It follows from Weyl's theory of equidistribution [2] that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)\,dx.$$

The purpose of this paper is to determine what modifications of this result are required when $f(x)$ is improperly Riemann integrable.

Every positive irrational number α has an infinite continued fraction expansion,

$$\alpha = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}},$$

where the b_i are integers such that $b_0 \geq 0$, and $b_i > 0$ for $i = 1, 2, 3, \cdots$. Let p_i/q_i $(i = 0, 1, 2, \cdots)$ be the convergents to α. The integers p_i and q_i are relatively prime, and

$$p_0 = b_0, \qquad q_0 = 1,$$
$$p_1 = b_1 b_0 + 1, \qquad q_1 = b_1,$$
$$p_{i+1} = b_{i+1} p_i + p_{i-1},$$
$$q_{i+1} = b_{i+1} q_i + q_{i-1},$$
$$p_i q_{i-1} - q_i p_{i-1} = (-1)^{i-1} \qquad (i = 1, 2, 3, \cdots).$$

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