A NOTE ON A PAPER BY L. C. YOUNG

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1. Introduction. Suppose that $f(x)$ is a real- or complex-valued function defined for all real x. For $0 \leq \alpha \leq 1$, we define the α-variation of $f(x)$ over $a \leq x \leq b$ as the least upper bound of the sums

$$\{\sum |\Delta f|^{1/\alpha}\}^{\alpha}$$

taken over all finite subdivisions of $a \leq x \leq b$. (When $\alpha = 0$, we denote by the above sum simply the maximum $|\Delta f|$.) We say that $f(x)$ is in W_α if it has finite α-variation over the interval $0 \leq x \leq 1$. L. C. Young has proved the following result.

THEOREM 1. (See [2, Theorem 4.2].) *Suppose that* $0 < \beta < 1$ *and that* $f(x)$, *with period* 1, *satisfies the condition*

$$\int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}|\, dt \leq h^{\beta} \qquad (h \geq 0)$$

for every monotone function $\phi(t)$ *such that*

$$\phi(t+1) = \phi(t) + 1$$

for all t. Then $f(x)$ *is in* W_α *for each* $\alpha < \beta$.

Young's argument does not suggest whether we can assert that $f(x)$ is in W_β. We present here an elementary proof for Theorem 1 and an example to show that this result is the best possible one in this direction.

2. Lemma. We require the following:

LEMMA 2. *Suppose that* $a_1, a_2, \cdots, a_N$ *and* $b_1, b_2, \cdots, b_N$ *are two sets of nonnegative numbers such that* $a_1 \geq a_2 \geq \cdots \geq a_N$ *and such that*

$$\sum_{\nu=1}^{n} a_\nu \leq \sum_{\nu=1}^{n} b_\nu$$

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