ON GROUPS OF ORTHONORMAL FUNCTIONS (II)

N. J. Fine

An orthonormal group on a measure-space Ω is defined as an orthonormal system of functions which is simultaneously a group with respect to multiplication. In an earlier note [1] we showed that, essentially, all such systems are derived from character groups of compact abelian groups.[1] It may be of interest to know, for a given Ω, what orthonormal groups it can support. The answer to this question for $\Omega = I$, the unit interval, is given by the following theorem.

THEOREM. *Let G be any countable abelian group. Then there exists on the unit interval I an orthonormal group G' isomorphic to G. If G is infinite, then G' is complete in $L^2(I)$.*

Proof. Assign the discrete topology to G, and let H be its character group. If G is of finite order n, then H and G are isomorphic. To each $h_k \in H$, $k = 1, 2, \cdots, n$, we associate the interval $I_k = [(k-1)/n, k/n)$, and define the n functions f_j by

$$f_j(x) = h_k(g_j) \qquad (x \in I_k) ,$$

where g_j are the elements of G. Then $\{f_j\}$ is the required orthonormal group.

If G is infinite, H is uncountable. The measure-algebra of H (with respect to normalized Haar measure) is non-atomic, separable, and normalized. Hence [2, p. 173] it is isomorphic to the measure-algebra of I. Now H is a complete separable metric space, and the outer Haar measure is a regular Caratheodory outer measure; the same is true of Lebesgue measure on I. Therefore we can apply a theorem of von Neumann [3, Th. 1] to obtain a measure-preserving transformation from H to I. The characters of H, transferred to I, then form the required orthonormal group, complete in $L^2(I)$.

[1] Only the case of a countable orthonormal group was considered in [1], but the proofs carry over to the uncountable case with slight modification.

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