ON GROUPS OF ORTHONORMAL FUNCTIONS (I)

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1. Introduction. Recently Civin [3, 4] and Chrestenson [2] have considered three specific systems of orthornormal functions on the unit interval which form multiplicative groups. They have shown that (subject to further restrictions) these systems are essentially characterized by their group structure. In this paper we propose to remove the topological restrictions on the base space and the group-theoretic restrictions on the system of functions.

Let $(\Omega, \mathfrak{Z}, m)$ be an abstract measure space[1], with m a countably-additive measure defined on the σ-ring $\mathfrak{Z}$, and $m(\Omega) = 1$. We may, and shall, assume that m is complete. Let

$$F = \{f_\alpha\} \qquad (\alpha = 0, 1, 2, \cdots)$$

be a family of complex-valued measurable functions on Ω, satisfying

$$(1) \qquad \int_\Omega f_\alpha \bar{f}_\beta \, dm = \delta_{\alpha\beta} \qquad (\alpha, \beta \geq 0),$$

$$(2) \qquad f_\alpha \bar{f}_\beta \in F \qquad (\alpha, \beta \geq 0).$$

We shall prove the following theorem:

THEOREM 1. *If $(\Omega, \mathfrak{Z}, m)$ and F are as above, then there exists a (unique) compact Abelian group H, satisfying the second axiom of countability, and a transformation T defined almost everywhere on Ω into H, such that*

(3) *the outer ν-measure of $Z = T(\Omega)$ is 1, and Z is dense in H, where ν denotes the Haar measure on H with $\nu(H) = 1$;*

(4) *for every ν-measurable set $M \subset H$, $T^{-1}(M) \in \mathfrak{Z}$ and $m(T^{-1}(M)) = \nu(M)$;*

[1] For the general measure- and group-theoretic concepts considered here, see [6] and [7].

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