FLOWS AND NONCOMMUTING PROJECTIONS ON HILBERT SPACE

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1. Introduction. Let $\{E(A)\}$ over $A \in \mathcal{B}_1$, the Borel subsets of the real line R_1, be a resolution of the identity for the Hilbert space X, and consider the flow

$$u_t = U_t u_0 = {}^{L}\!\!\int_{-\infty}^{\infty} e^{it\lambda} dE(\lambda)\, u_0 ,$$

over t real for fixed $u_0 \in X$. Let P be an orthogonal projection in X. Our problem is to study the asymptotic behavior of $||Pu_t||^2$ as $t \longrightarrow +\infty$ or $t \longrightarrow -\infty$. If P commutes with all $E(A)$, then

$$PU_t = U_t P \quad \text{and} \quad ||Pu_t||^2 = ||Pu_0||^2,$$

a constant, so we are interested only in the case where P fails to commute.

It is easy to see that this asymptotic behavior depends upon the nature of γ through the equation

$$||Pu_t||^2 = \int_{R_2} e^{it(x-y)} d\gamma(x,y)$$

integrated in a Riemann sense over the plane R_2, where

$$\gamma(A \times B) = (PE(A)u_0,\ E(B)u_0).$$

If γ admits a σ-additive and bounded extension over $\mathcal{B}_2$, the Borel sets of R_2, standard procedures enable us to say that $||Pu_t||^2$ converges densely to C as $t \longrightarrow +\infty$ or $t \longrightarrow -\infty$ if and only if $\gamma(D_s) = 0$ for $s \neq 0$ and $\gamma(D_0) = C$, where the diagonal

$$D_s = \{(x, y) \in R_2 \mid x - y = s\}.$$

The interesting fact here is, as we shall see by example, that γ need not in general be either σ-additive or bounded, although it is always both if P is

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