A CHARACTERIZATION OF COMPLETE LATTICES

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1. Introduction. A complete lattice $\mathfrak{A} = \langle A, \leq \rangle$ has the property that every increasing function on A to A has a fixpoint.[1] Tarski raised the question whether the converse of this result also holds. In this note we shall show that the answer to this question is affirmative, thus establishing a criterion for completeness of a lattice in terms of fixpoints.[2]

We shall use the notation of [**6**]. In addition, the formula $a \not\leq b$ will be used to express the fact that $a \leq b$ does not hold. By $\langle a_\xi ; \xi < \alpha \rangle$, where α is any (finite or transfinite) ordinal we shall denote the sequence whose consecutive terms are $a_0, a_1, \cdots, a_\xi, \cdots$ (with $\xi < \alpha$); the set of all terms of this sequence will be denoted by $\{a_\xi ; \xi < \alpha\}$. The sequence $\langle a_\xi ; \xi < \alpha \rangle$ is, of course, called *increasing*, or *strictly increasing*, if $a_\xi \leq a_{\xi'}$, or $a_\xi < a_{\xi'}$, for any $\xi < \xi' < \alpha$; analogously we define *decreasing* and *strictly decreasing* sequences.

2. A lemma. We start with the following:

LEMMA 1. *If the lattice* $\mathfrak{A} = \langle A, \leq \rangle$ *is incomplete, then there exist two sequences* $\langle b_\xi ; \xi < \beta \rangle$ *and* $\langle c_\eta ; \eta < \gamma \rangle$ *such that*

(i) $b_\xi < c_\eta$ *for every* $\xi < \beta$ *and every* $\eta < \gamma$,

(ii) $\langle b_\xi ; \xi < \beta \rangle$ *is strictly increasing and* $\langle c_\eta ; \eta < \gamma \rangle$ *is strictly decreasing,*

(iii) *there is no element* $a \in A$ *which is both an upper bound of* $\{b_\xi ; \xi < \beta\}$ *and a lower bound of* $\{c_\eta ; \eta < \gamma\}$.[3]

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[1] See [**6**] (where further historical references can also be found).

[2] This result was found in 1950 and outlined in [**2**].

[3] A related, though weaker, property of incomplete lattices is mentioned implicitly in [**1**, p. 53, Exercise 4].

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