AN EXTENSION OF A THEOREM OF JORDAN AND VON NEUMANN

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1. Introduction. Let $\{E\}$ denote the class of generalized euclidean spaces E (that is, $E \subseteq \{E\}$ provided all finite dimensional subspaces of E are euclidean spaces). The problem of characterizing metrically the class $\{E\}$ with respect to the class $\{B\}$ of all Banach spaces has been solved in many different ways.[1] Fréchet's characteristic conditions [5]

$$(*) \qquad \frac{1}{2}\sum_{i,j=1}^{3} [\,\|p_i\|^2 + \|p_j\|^2 - \|p_i - p_j\|^2]x_i x_j \geq 0, \qquad (p_1, p_2, p_3 \in \mathbf{B})$$

was immediately weakened by Jordan and von Neumann [6] to

$$(**) \qquad \|p_1 + p_2\|^2 + \|p_1 - p_2\|^2 = 2(\|p_1\|^2 + \|p_2\|^2) \qquad (p_1, p_2 \in \mathbf{B}).$$

This relation has now become a kind of standard to which others repair by showing that it is implied by newly postulated conditions [3, 4, 10], and it has been, apparently, the motivation of work in which it does not enter directly [7, 9]. Perhaps the best possible result in this direction, however, is due to Aronszajn [1] who assumed merely that

$$\|(x+y)/2\| = \frac{1}{2}\,\phi(\|x\|, \|y\|, \|x-y\| \qquad (x, y \in \mathbf{B}),$$

with ϕ *unrestricted* except for being nonnegative and $\phi(r, 0, r) = r$, $r \geq 0$.

These conditions, and others like them, are all equivalent in a Banach space, for each is necessary and sufficient to insure the euclidean character of all subspaces. In a more general environment, however, this is not the case, and so the desirability of making a comparative study of such conditions in more general spaces is suggested. In this note the larger environment is furnished by the class $\{M\}$ of complete, metrically convex and externally convex,

[1] This note deals exclusively with normed linear spaces over the field of reals.

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