SIMPLE PROOF OF A THEOREM OF P. KIRCHBERGER

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In their paper Rademacher and Schoenberg [2] give a simple proof of a theorem due to P. Kirchberger [1] on separation of sets in the Euclidean n-space by means of hyperplanes. Their proof utilizes Helly's theorem on convex sets and is one instance of several important applications of that theorem given in their paper.

The object of this note is to give a simple direct proof of Kirchberger's theorem. Our proof does not use Helly's theorem and is based on the well-known theorem of Carathéodory on convex sets (which is also utilized by Rademacher and Schoenberg in their derivation of Helly's theorem).

The statement and proof of the theorem follow.

THEOREM OF P. KIRCHBERGER. Let S and T be two finite sets of points in the Euclidean n-space, E^n. If any $k+1$ points of S may be separated from any $l+1$ points of T by a hyperplane of E^n ($k+l \leq n$), then S may be separated from T by a hyperplane of E^n.

Proof. We first prove our theorem under the (seemingly) stronger condition that the assumption of the theorem holds for any k, l satisfying $k \leq n$, $l \leq n$, instead of $k + l \leq n$.

It is well-known that two compact convex sets in E^n having no point in common may be separated by a hyperplane of E^n. The assertion of the theorem is therefore equivalent to the following: The convex hulls $H(S)$ and $H(T)$ of S and T respectively have no point in common. If this were not the case, any point of $H(S) \cap H(T)$ would, by Carathéodory's theorem, belong to two simplexes of dimensions not exceeding n with vertices in S, T respectively. But, according to our assumption, the vertices of the one simplex may be separated from those of the other by a hyperplane of E^n, and therefore their convex hulls may be similarly separated. The contradiction obtained proves the validity of our assertion.

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