A CONVERSE OF HELLY'S THEOREM ON CONVEX SETS

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1. Introduction. Helly's well known theorem on convex sets states that families of compact convex sets in Euclidean n-space E^n, have the following property:

Property $\mathfrak{H}$: *If every $n+1$ of the sets have a point in common, then there exists a point common to all sets of the family.*

If a family of compact sets in E^n has property $\mathfrak{H}$ this, clearly, does not imply that the sets are convex. The purpose of this short note is to show that (loosely speaking) if the family possesses property $\mathfrak{H}$ not accidentally but by virtue of the geometric structure of its sets, then all the sets of the family are convex. The proof of our result is rather simple, but as far as we are aware no theorems converse to Helly's have been noticed before.

In order to state our result briefly we make the following definition:

DEFINITION. A family of sets K_α in E^n is said to have property $\mathfrak{A}\mathfrak{H}$ if every family $\{K'_\alpha\}$, with $K'_\alpha = T_\alpha K_\alpha$ an affine[1] transform of K_α, possesses property $\mathfrak{H}$.

We may now formulate our result.

THEOREM. *Let $\{K_\alpha\}$ be a family of more than $n+1$ compact sets in E^n, all having linear dimension n (that is, no K_α lies in a hyperplane). If the family has property $\mathfrak{A}\mathfrak{H}$ then all sets K_α are convex.*

2. Proof. We shall show that if one of the sets of the family, say K_0, is not convex then the family cannot have property $\mathfrak{A}\mathfrak{H}$.

Since K_0 is closed and its linear dimension is n, there exist $n+1$ points $P_1, \cdots, P_{n+1} \in K_0$ forming the vertices of a simplex whose interior contains points not belonging to K_0. Let P_0 be such a point.

[1] By an affine transformation we understand a nonsingular one.

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