ON EIGENVALUES OF SUMS OF NORMAL MATRICES

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1. Problem, notations, results. A well-known theorem due essentially to Bendixson [1, Theorem II] states that if X and Y are hermitian $n \times n$ matrices with eigenvalues

$$\xi_1 \leq \xi_2 \leq \cdots \leq \xi_n \quad \text{and} \quad \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n ,$$

then every eigenvalue λ of $X + iY$ is contained in the rectangle

$$\xi_1 \leq \Re\lambda \leq \xi_n, \quad \eta_1 \leq \Im\lambda \leq \eta_n .$$

What is the exact range of λ, for given ξ_ν and η_ν? We shall solve the following slightly more general problem, referring to normal instead of hermitian matrices. Let $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ *be given complex numbers. Describe geometrically the set* Λ *of all numbers* λ *which may occur as eigenvalues of* $A + B$, *where* A *and* B *run over all normal* $n \times n$ *matrices with eigenvalues* $\alpha_1, \cdots, \alpha_n$ *and* $\beta_1, \cdots, \beta_n$ *respectively.*

To state the results concisely let us denote by the terms *circular region* and *hyperbolic region* every set of complex numbers $\xi + i\eta$ which may be described, using some real constants a, b, c, d, by

$$a\xi + b\eta + c(\xi^2 \pm \eta^2) + d \geq 0. \tag{1}$$

where $+$ refers to the circular, $-$ to the hyperbolic case. We denote by $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ the sets whose elements are $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ respectively. For every two sets Γ, Δ of complex numbers we denote by $\Gamma + \Delta$ the set whose elements are all $\gamma + \delta$, where $\gamma \in \Gamma$, $\delta \in \Delta$. Our main result is

THEOREM 1. *If* $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ *are arbitrary complex numbers, then the set* Λ *defined above can be represented as an intersection*:

$$\Lambda = \bigcap_{\Gamma} (\{\alpha_\nu\} + \Gamma)$$

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