THE MEASURE RING FOR A CUBE OF ARBITRARY DIMENSION

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1. Introduction. From Maharam's theorem [2] on the structure of measure algebras it is very easy to obtain a unique characterization, in terms of cardinal numbers, of an arbitrary measure ring. An example of a measure ring is the ring of Hellinger types of (finite) measures on an additive class of sets. In this note, the cardinal numbers are computed which characterize the ring of Hellinger types of measures on the Baire subsets of a cube of arbitrary dimension.

2. Definitions. A lattice R is a *Boolean σ-ring* if (a) the family R_x of all subelements of any given element x form a Boolean algebra, and (b) any countable family of elements has a least upper bound. If, in addition, for each x in R there exists some countably additive finite-valued real function on R_x which is 0 only for the zero-element of R, then R is a *measure ring.*[1] A Boolean σ-ring with a largest element is a *Boolean σ-algebra.* A measure ring with a largest element is a *measure algebra.* The measure algebra of a finite measure μ is the Boolean σ-algebra of μ measurable sets modulo μ null sets.

A subset S of a Boolean σ-algebra R is a *σ-basis* if the smallest σ-subalgebra of R containing S is R itself. R is *homogeneous of order* α if, for every nonzero element x of R, the smallest cardinal number of a σ-basis of R_x is α. We observe that R cannot be homogeneous of finite nonzero order. If it is homogeneous of order 0, it is the two-element Boolean algebra.

Let α be an infinite cardinal, I the unit interval $[0, 1]$, and I^α the topological product of I with itself α times (the α-dimensional cube). $L^{(\alpha)}$ will denote the product Lebesgue measure on the Baire subsets of I^α, and $M^{(\alpha)}$ the measure algebra of $L^{(\alpha)}$. Then it is not hard to see that $M^{(\alpha)}$ is homogeneous of order α. Maharam has in fact shown that it is, essentially, the only measure algebra of order α.

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[1] Our use of the terms 'measure ring', 'measure algebra', unlike Maharam's, refers only to the algebraic structure. In this sense two measure rings are isomorphic if they are isomorphic as σ-rings.

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