A SUM CONNECTED WITH THE SERIES FOR THE PARTITION FUNCTION

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1. Introduction. The famous formula of Rademacher [5] for the number $p(n)$ of partitions of an integer n states that

$$p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}A_k(n)k^{1/2}\frac{d}{dn}\left(\frac{\sinh(K\lambda/k)}{\lambda}\right),$$

where $K=\pi(2/3)^{1/2}$, $\lambda=(n-1/24)^{1/2}$ and the series is absolutely convergent. The coefficients $A_k(n)$ are defined by

$$A_1(n)=1\ ,\quad A_2(n)=(-1)^n\ ,\quad A_3(n)=2\cos[\pi(12n-1)/18]\ ,$$

and in general

$$A_k(n)=\sum_{(h,k)=1}\omega_{h,k}\exp(-2\pi ihn/k)\ , \tag{1.1}$$

where h ranges over those numbers which are less than k and prime to k. The numbers $\omega_{h,k}$ are certain $24k$th roots of unity which arise in the theory of modular functions and are defined by

$$\omega_{h,k}=\left(\frac{-h}{k}\right)\exp\left[-\left\{\frac{1}{4}(k-1)+\frac{1}{12}\left(k-\frac{1}{k}\right)(2h+\bar{h}-h^2\bar{h})\right\}\pi i\right] \tag{1.2}$$

if k is odd, and by

$$\omega_{h,k}=\left(\frac{-k}{h}\right)\exp\left[-\left\{\frac{1}{4}(2-hk-h)+\frac{1}{12}\left(k-\frac{1}{k}\right)(2h+\bar{h}-h^2\bar{h})\right\}\pi i\right] \tag{1.3}$$

when k is even. Here $(a|b)$ is the symbol of Jacobi and $\bar{h}$ is defined as any solution of the congruence $h\bar{h}\equiv 1 \pmod{k}$.

Because of the intricacy of the numbers $\omega_{h,k}$ the task of evaluating $A_k(n)$ for large k directly from its definition in (1.1) is quite formidable. To surmount this difficulty D. H. Lehmer [3] made an intensive study of the $A_k(n)$. He was able to reduce them to sums studied by H. D. Kloosterman and H. Salié. In the first place he factored the $A_k(n)$ according to the prime number powers contained in k. Secondly, by using Salié's formulas, he evaluated $A_k(n)$ explicitly in the case in which k is a prime or a power of a prime. Both results together provide a method for calculating the $A_k(n)$. It should also be mentioned that another

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