BOUNDEDNESS IN TOPOLOGICAL RINGS

EDWIN WEISS

Introduction. The purpose of this note is to dispose of certain preliminaries (and of some peripheral remarks) in the direction of a structure theory of the Wedderburn-Artin-Jacobson type for a rather restricted class of topological rings—namely, bounded ones. The notion of boundedness, which may be looked on as an algebraic analogue of compactness, was introduced by Shafarevich [6] and later considered by Kaplansky [3]. It is not unexpected that in an algebraic approach to the study of topological rings the concept of boundedness should prove fruitful; where, by an algebraic approach is meant one in which the use of deep topological facts is avoided—thus, for example, we shall not use any results about the structure of locally compact groups, since such results depend on Haar measure, the Peter-Weyl theorem and Pontrjagin duality.

Since the study of the radical is one of the foundation stones of the classical structure theory of rings, and in view of our self-imposed restrictions on available techniques, it is natural to attempt to extend the notion of radical in such a way as to take the topology of the ring into account. Such an attempt is the primary concern of this note. The proofs will often be merely slight extensions of the standard ones for discrete rings.

1. Definitions and preliminaries. As usual, (see, for example, [3]) by a *topological ring* we mean a set R which is a ring and a Hausdorff space and such that the mappings $(x, y) \to x-y$ and $(x, y) \to xy$ of $R \times R \to R$ are both continuous. A subset S of R is *left bounded* if for any neighborhood U of 0 there exists a neighborhood V of 0 (V depends on U) such that $V \cdot S \subset U$, where $V \cdot S = \{xy | x \in V, y \in S\}$. Right boundedness is defined in an analogous way. We say that S is *bounded* if it is both left and right bounded. It is clear that a subset S of R is bounded if and only if, for any neighborhood U of 0, there exists a neighborhood V of 0 such that $V \cdot S \cdot V \subset U$. If the set R itself is bounded, we say that R is a *bounded ring*.

Let M be a left R-module; M is called a *topological left R-module* when: R is a topological ring, M is a topological group (this includes Hausdorff), and the map $(\alpha, x) \to \alpha x$ of $R \times M \to M$ is continuous. Similarly, the notion of topological right R-module is defined. Since there is no essential distinction between right and left, we shall usually state things only for topological left R-modules.

Received November 26, 1954. This paper was written while the author was an NSF post-doctoral fellow.