THE SYMMETRY FUNCTION IN A CONVEX BODY

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Let K_n be an n-dimensional convex body in n-dimensional Euclidean space E_n. At each point P in K_n consider the largest subset $S(P)$ of K_n radially symmetric with respect to the point P. This set is well-defined and convex for it is simply the intersection of K_n with its radial reflection through the point P. Let $m(P)$ equal the measure of $S(P)$ and let $f(P)$ equal $m(P)V_n^{-1}$ where V_n is the measure of K_n. Clearly $0 \leq f(P) \leq 1$ for all P in K_n and $f(P)=0$ only if P is on the boundary of K_n; also f is continuous. Moreover f attains the value 1 only if K_n is radially symmetric. The object of this note is to present various properties of this function f.

THEOREM 1. (Besicovitch [1], $n=2$). *There is a point P in K_2 such that* $f(P)=2/3$. (In [3, p. 46] this theorem is ascribed to S. S. Konvyer.)

THEOREM 2. (Besicovitch [2], $n=2$). *If K_2 is of constant width then there is a point P in K_2 such that* $f(P)=.840\cdots$.

H. G. Eggleston [4] studied further the symmetric function in a body of constant width.

Using a result of P. C. Hammer [5] on the ratio which the centroid of a convex body divides the chords passing through it, F. W. Levi [6] obtained the following.

THEOREM 3. *If P is the centroid of K_n then*

$$f(P) \geq 2(1+n^n)^{-1} .$$

The following properties of f will be obtained.

THEOREM 4. $\int_{K_n} f = 2^{-n} V_n$.

COROLLARY. *There is a point P in K_n such that $f(P) > 2^{-n}$.*

THEOREM 5. *If a is a real number then the set of points P in K_n at which $f(P) \geq a$ is convex. Furthermore f attains its maximum value at precisely one point.*

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