ON THE NUMBER OF SINGULAR POINTS, LOCATED ON THE UNIT CIRCLE, OF CERTAIN FUNCTIONS REPRESENTED BY C-FRACTIONS

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1. The continued fraction

$$(1) \qquad 1+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{d_n z^{\alpha_n}}{1}\right),$$

where the α_n are positive integers and the $d_n \neq 0$ for all $n \geqq 1$, are sometimes called C-fractions. They were first studied by Leighton and Scott [2]. It is well known (see [1]) that if

$$(2) \qquad \lim_{n\to\infty} |d_n|^{1/\alpha_n}=1 \quad \text{and} \quad \lim_{n\to\infty} \alpha_n=\infty ,$$

the continued fraction (1) converges to a function which is meromorphic for all $|z|<1$. The first results concerning the location of singular points of functions of this type were obtained by Scott and Wall [4]. Considerably better results were recently obtained by one of the present authors [5, 6]. In all of these results the continued fractions are assumed to satisfy the conditions (2) or even more restrictive ones. In this paper we are able to weaken condition (2) and replace it by

$$(2)' \qquad \begin{aligned} &(\mathrm{a}) \quad \lim_{n\to\infty} (4|d_n|)^{1/\alpha_n}=1 , \\ &(\mathrm{b}) \quad \text{there exists a sequence } \{\alpha_{n_k}\} \text{ such that} \\ &\qquad \lim_{k\to\infty} \alpha_{n_k}=\infty \quad \text{and} \quad \lim_{k\to\infty} n_k/k<2 . \end{aligned}$$

While all but one of the previous results gave sufficient conditions for the function represented by (1) to have the circle $|z|=1$ as a natural boundary, we give here criteria which are sufficient in order that the function has at least p singular points on the circle.

Let $A_n(z)/B_n(z)$ be the nth approximant of (1) and let σ_n and τ_n be the degrees of the polynomials $A_n(z)$ and $B_n(z)$, respectively. Also, let ρ_n be the maximum of the degrees of $A_n^*(z)$ and $B_n^*(z)$ where $A_n^*(z)/B_n^*(z)$ is the nth approximant of (1) when the d_n are replaced by their moduli. Then

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