NOTE ON THE LERCH ZETA FUNCTION

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1. Introduction. The functional equation for the Riemann zeta function and Hurwitz's series were derived by a formal application of Poisson's summation formula in some previous papers [9], [10], [12, p. 17]. The results are correct although obtained by equating two infinite series whose regions of convergence exclude each other. It was shown later that this difficulty can be overcome if a generalized form of Poisson's formula is used [6]. It is the purpose of this note to show that the application of Poisson's summation formula in its ordinary form to the more general case of Lerch's zeta function [1], [2], [3], [5, pp. 27–30], [8] does not present the difficulties with respect to convergence which arise in its special cases. Thus a very simple and direct method for developing the theory of this function can be given. It may be noted that Hurwitz's series can be obtained immediately upon specializing one of the parameters. (cf. also [1]).

2. Lerch's zeta function $\Phi(z, s, v)$ is usually defined by the power series

(1a) $$\Phi(z, s, v)=\sum_{m=0}^{\infty} z^m(v+m)^{-s} \qquad |z|<1,\ v\neq 0, -1, -2, \cdots.$$

Equivalent to this is the integral representation [5, pp. 27–30]

(1b) $$\Phi(z, s, v)=\frac{1}{2}v^{-s}+\int_0^\infty (v+t)^{-s}z^t dt$$
$$+2\int_0^\infty \sin\left(s\tan^{-1}\frac{t}{v}-t\log z\right)(v^2+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}dt$$
$$|z|<1,\ \mathscr{R}v>0,$$

which can be written as

(1c) $$\Phi(z, s, v)=\frac{1}{2}v^{-s}+z^{-v}\Gamma(1-s)\left(\log\frac{1}{z}\right)^{s-1}-z^{-v}v^{1-s}\sum_{n=0}^{\infty}\frac{\left(-v\log\frac{1}{z}\right)^n}{n!\,(n+1-s)}$$
$$+2\int_0^\infty \sin\left(s\tan^{-1}\frac{t}{v}-t\log z\right)(v^2+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}dt$$
$$z\neq 1,\ \mathscr{R}v>0$$

upon replacing the first integral on the right of (1b) by the second and

Received February 11, 1954.