LIMITS OF RATIONAL FUNCTIONS

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Numerous results relating the location of the zeros of a sequence of polynomials to the form of possible limit functions of the sequence are known. These results are due in the main to Laguerre, Lindwart, Pólya, and Korevaar. Summaries and references are to be found in [5] and [1]. For example, the following is a theorem of Lindwart and Pólya [3]. *Let $P_n(z)$ be a sequence of polynomials with all zeros real and let $P_n(z)$ converge uniformly in some domain to a function no identically zero. Then $P_n(z)$ converges, uniformly in every compact subset of the plane, to an entire function of the form $e^{-cz^2}f(z)$, where c is a constant $\geqq 0$ and $f(z)$ is of genus $\leqq 1$.*

We shall show, in Theorems 2 and 3, that the state of affairs is violently altered if instead of polynomials we consider rational functions with real zeros and real poles. Essentially, the convergence and-or non-convergence can be anything compatible with the fact that no limiting function, not identically zero, can have a non-real zero.

Various theorems of Saxer, Montel, and Obrechkoff specify the possible form of the limit of a sequence of rational functions. A résumé and references are contained in Obrechkoff [5]. All of these results depend on conditions on the rational functions involving either the location of the poles relative to the zeros or the behavior of expressions involving poles and residues.

The proof of Theorems 2 and 3 hinges on the fact that if $f(z)$ is holomorphic and $\neq 0$ in $\mathscr{I}z>0$, then there exists a sequence of rational functions $R_n(z)$ with real zeros and poles such that $R_n(z)\to f(z)$ uniformly in every compact subset of $\mathscr{I}z>0$. This is a special case of Theorem 1 below, which is similar to a previous result of ours for polynomials [4].

THEOREM 1. *Let Γ be a rectifiable Jordan curve on the z-sphere and let D be one of the two domains determined by Γ. Let $f(z)$ be holomorphic and $\neq 0$ in D. Then there exists a sequence of rational functions $R_n(z)$, $n\geqq 1$, such that all zeros and poles of $R_n(z)$ are on Γ and $R_n(z)\to f(z)$ uniformly in every compact subset of D.*

Note. If $\infty\notin\Gamma$, then each $R_n(z)$ is of the form $P(z)/Q(z)$, where P and Q are of the same degree and have zeros only on Γ. For $\infty\in\Gamma$, P and Q may have different degrees.

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