SUMMABLE TRIGONOMETRIC SERIES

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1. Introduction. One of the problems in the theory of trigonometric series in the form

$$(1.1) \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} a_n(x)$$

is that of suitably defining a process of integration such that, if the series (1.1) converges to a function $f(x)$, then $f(x)$ is integrable and the coefficients a_n, b_n are given in Fourier form. The problem has been solved by Denjoy [3], Verblunsky [10], Marcinkiewicz and Zygmund [8], Burkill [1], [2], and James [6]. In Verblunsky's paper and in Burkill's first paper, additional hypotheses other than the convergence of (1.1) were made, and in all the papers some modification of the form of the Fourier formulas was necessary:

An extension of the problem is to consider series that are summable (C, k), $k \geqq 1$. This has been solved by Wolf [11] when the sum function is Perron integrable. The problem of defining a process of integration which may be applied to any series summable (C, k) may be solved if an additional condition involving the conjugate series

$$(1.2) \qquad \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\infty} b_n(x)$$

is imposed. With this extra condition, it is proved, in § 2, that the formal product of $\cos px$ or $\sin px$ and a series summable (C, k) to $f(x)$ is also summable (C, k) to $f(x)\cos px$ or $f(x)\sin px$.

In § 3, some properties of integrated series are discussed and then, in § 4, it is shown that the generalized P^{k+2}-integral [7] integrates any trigonometric series summable (C, k) and satisfying the extra condition. In addition, the coefficients are given by a natural modification of the Fourier formulas. These are the principal results of the paper. They were described briefly for the special case $k=2$ in the author's invited address at the 1954 Summer Meeting of the American Mathematical Society.

It is also possible to improve the results slightly and only require summability for all x in $[0, 2\pi]$ with the possible exception of a countable set. This requires a minor modification in the definition of the P^{k+2}-integral and these changes are indicated in §§ 5 and 6.

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