A NOTE ON ORTHOGONAL SYSTEMS

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1. Let $\omega_n(x)$, $n=1, 2, \cdots$, $0 \leqq x \leqq 1$, be an orthonormal set of functions which are uniformly bounded,

$$(1) \qquad |\omega_n(x)| \leqq M \qquad (n=1, 2, \cdots, 0 \leqq x \leqq 1).$$

If $\sum_1^\infty |a_n| < \infty$, and if $\int_0^1 |g(x)| dx < \infty$ we may define

$$(2) \qquad f(x) = \sum_{n=1}^\infty a_n \omega_n(x), \qquad b_n = \int_0^1 g(x)\overline{\omega_n(x)} dx.$$

The following inequalities were established by R. E. A. C. Paley [1]:

$$(3) \qquad \begin{aligned}
&\left[\int_0^1 |f(x)|^q dx\right]^{1/q} \leqq A_1'(q)\left[\sum_{n=1}^\infty |a_n|^q n^{q-2}\right]^{1/q} && (2 \leqq q < \infty);\\
&\left[\sum_{n=1}^\infty |b_n|^p n^{p-2}\right]^{1/p} \leqq A_2'(p)\left[\int_0^1 |g(x)|^p dx\right]^{1/p} && (1 < p \leqq 2);\\
&\left[\int_0^1 |f(x)|^p x^{p-2} dx\right]^{1/p} \leqq A_3'(p)\left[\sum_{n=1}^\infty |a_n|^p\right]^{1/p} && (1 < p \leqq 2);\\
&\left[\sum_{n=1}^\infty |b_n|^q\right]^{1/q} \leqq A_4'(q)\left[\int_0^1 |g(x)|^q x^{q-2} dx\right]^{1/q} && (2 \leqq q < \infty).
\end{aligned}$$

In the present paper we shall establish some related results which are however a great deal simpler. We shall prove that

$$(4') \qquad \left[\int_0^1 |f(x)|^2 x^{-2\alpha} dx\right]^{1/2} \leqq A_1(\alpha)\left[\sum_{n=1}^\infty |a_n|^2 n^{2\alpha}\right]^{1/2} \qquad (0 \leqq \alpha < 1/2);$$

$$(4'') \qquad \left[\sum_{n=1}^\infty |b_n|^2 n^{-2\alpha}\right]^{1/2} \leqq A_2(\alpha)\left[\int_0^1 |g(x)|^2 x^{2\alpha} dx\right]^{1/2} \qquad (0 \leqq \alpha < 1/2).$$

As Paley pointed out, the inequalities (3) include the inequalities of F. Riesz which assert that

$$(5) \qquad \begin{aligned}
&\left[\int_0^1 |f(x)|^q dx\right]^{1/q} \leqq B(p)\left[\sum_{n=1}^\infty |a_n|^p\right]^{1/p},\\
&\left[\sum_{n=1}^\infty |b_n|^q\right]^{1/q} \leqq B(p)\left[\int_0^1 |g(x)|^p dx\right]^{1/p} \qquad (1 \leqq p \leqq 2,\ 1/p+1/q=1).
\end{aligned}$$

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