CONSTRUCTIVE PROOF OF THE MIN-MAX THEOREM

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1. Introduction. The foundations of a mathematical theory of "games of strategy" were laid by John von Neumann between 1928 and 1941.[1] The publication in 1944 of the book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern climaxed this pioneering effort. The first part of this volume is concerned with games with a finite number of pure strategies with particular emphasis on the "zero-sum two-person" type of game. There it is shown that in most instances a player is at a disadvantage if he always plays the same pure strategy and that it is better to "mix" his pure strategies by some chance device. The starting point of all discussions of this type of game is the celebrated "Main Theorem" or Min-Max Theorem which is concerned with existence and properties of optimal mixed strategies for both players.

The first proofs of this theorem, given by von Neumann, made rather involved use of topology, functional calculus, and fixed point theorems of L. E. J. Brouwer. The first proof of an elementary character was given by J. Ville, 1938. The von Neumann-Morgenstern book, with the purpose of having a proof which is accessible to a less highly trained group, carries the theme of elementarization further [6]. At this late date there still continues to be a need for a truly elementary proof; for example, the recent book of McKinsey on game theory [5] omitted a self-contained proof because none was available.

Kuhn [4] gives a bibliography of some of the better known proofs of the Min-Max Theorem, together with a discussion of their general characteristics which he broadly classifies into (1) those based on separation properties of convex sets and (2) those using some notion of a fixed point of a transformation. Kuhn [4] and McKinsey [5] provide proofs along the lines of von Neumann [6] based on a separation theorem. Dresher [3] gives a self-contained proof along the lines of Ville. As was pointed out in [7], the Min-Max Theorem is completely algebraic and should be given an algebraic proof. The purely algebraic proofs, when made self-contained and elementary, appear to be quite long, [3], [4], [7], and, with the exception of Weyl's proof [7], make use of nonalgebraic concepts as the minimum of a continuous function on a closed bounded set is assumed on the set. All these proofs are either pure existence proofs

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[1] For the contributions of Borel to this field see *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 95-127.