A NON-ARCHIMEDIAN MEASURE IN THE SPACE OF REAL SEQUENCES

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1. Introduction. Let S be the set of real sequences $X=(x_n)$. For $X, Y \in S$ we define $X+Y=(x_n+y_n)$, 0 as the sequence $x_n=0$ and introduce order by writing $X>0$ when for some m, $x_n=0$ for $n<m$ and $x_m>0$. Thus S may be considered as an ordered abelian group with a non-archimedian order. Let S be topologized by considering the open intervals

$$(X, Y)=\{Z|X<Z<Y\}$$

as a basis for the open sets. Then S is a topological group. We note that S is not locally compact. We wish to define a measure on S which is invariant with respect to translations of measurable sets by elements in S and which assigns a nonzero measure to the sets in a basis for the topology in S. It is evident from a consideration of the spheres in Hilbert space that such a measure can not in general be real valued for spaces which are not locally compact. In the example studied here the range of the measure function is a subset of S.

The ring of measurable sets which serves as the domain of the measure function is generated by a class of sets called intervals. We shall show that these intervals are a basis for the topology of S defined by the open intervals. They have some properties of the real half-open intervals $a' \leq x < a''$ which are useful in deriving the properties of a measure function.

For a positive integer p and real numbers

$$a_1, \cdots, a_{p-1}, a'_p, a''_p$$

let $I_p=I(a_1, \cdots, a_{p-1}; a'_p, a''_p)$ be the set of $X=(x_n) \in S$ such that

$$x_n=a_n, \qquad \text{for } n<p,$$

$$a'_p \leqq x_p < a''_p$$

$$-\infty < x_n < +\infty, \qquad n>p.$$

If $p=1$ there are no conditions on the x_n for $n<p$. If $a''_p \leqq a'_p$ then I_p is empty. That the sets I_p and the open intervals (X, Y) are equivalent as bases for neighborhood topologies is shown as follows:

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