AN ANALOG OF THE MINIMAX THEOREM FOR VECTOR PAYOFFS

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1. Introduction. The von Neumann minimax theorem [2] for finite games asserts that for every $r \times s$ matrix $M=\|m(i, j)\|$ with real elements there exist a number v and vectors

$$p=(p_1, \cdots, p_r), \; q=(q_1, \cdots, q_s), \quad p_i, q_j \geqq 0, \; \sum p_i = \sum q_j = 1$$

such that

$$\sum_i p_i m(i, j) \geqq v \geqq \sum_j q_j m(i, j)$$

for all i, j. Thus in the (two-person, zero-sum) game with matrix M, player I has a strategy insuring an expected gain of at least v, and player II has a strategy insuring an expected loss of at most v. An alternative statement, which follows from the von Neumann theorem and an appropriate law of large numbers is that, for any $\varepsilon > 0$, I can, in a long series of plays of the game with matrix M, guarantee, with probability approaching 1 as the number of plays becomes infinite, that his average actual gain per play exceeds $v-\varepsilon$ and that II can similarly restrict his average actual loss to $v+\varepsilon$. These facts are assertions about the extent to which each player can control the center of gravity of the actual payoffs in a long series of plays. In this paper we investigate the extent to which this center of gravity can be controlled by the players for the case of matrices M whose elements $m(i, j)$ are points of N-space. Roughly, we seek to answer the following question. Given a matrix M and a set S in N-space, can I guarantee that the center of gravity of the payoffs in a long series of plays is in or arbitrarily near S, with probability approaching 1 as the number of plays becomes infinite? The question is formulated more precisely below, and a complete solution is given in two cases: the case $N=1$ and the case of convex S.

Let

$$M=\|m(i, j)\|, \qquad 1 \leqq i \leqq r, \; 1 \leqq j \leqq s$$

be an $r \times s$ matrix, each element of which is a probability distribution over a closed bounded convex set X in Euclidean N-space. By a *strategy for Player* I is meant a sequence $f=\{f_n\}$, $n=0, 1, 2, \cdots$ of functions, where f_n is defined on the set of n-tuples $(x_1, \cdots, x_n)$, $x_i \in X$

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