A RELATION BETWEEN PERFECT SEPARABILITY, COMPLETENESS, AND NORMALITY IN SEMI-METRIC SPACES

LOUIS F. MCAULEY

1. Introduction. This paper proves that a regular semi-metric[1] topological space S may have such properties as hereditary separability, collectionwise normality [1], paracompactness [10], and weak completeness without being either a developable space [1] or a metric space. However, if S is strongly complete, then hereditary separability implies perfect separability [12] and consequently metrizability. It has been proved [1; 12] that a regular developable topological space (Moore space) is metrizable provided that it is perfectly separable. Thus, a regular semi-metric topological space may be far removed from a Moore space contrary to a result announced by C. W. Vickery [11]. The notion of p-separability due to Frechet is generalized and a question raised by W. A. Wilson [14, p. 336] is answered in the affirmative. Throughout this paper, S denotes a regular semi-metric topological space.

2. Weak and strong completeness.

DEFINITION 2.1. A space S is said to be $\left\{\begin{matrix}\text{weakly complete}\\ \text{strongly complete}\end{matrix}\right\}$ provided there exists a distance function d such that (1) the topology of S is invariant with respect to d and (2) if $\{M_i\}$ is a monotonic decending sequence of closed subsets of S such that, for each i, there exists a $1/i$-neighborhood of a point p_i $\left\{\begin{matrix}\text{in } M_i\\ \text{in } S\end{matrix}\right\}$ which contains M_i, then ΠM_i contains a point.

It is now shown that strong completness is sufficient to bridge a gap between a hereditarily separable space S and a developable space.

Received August 13, 1954 and in revised form April 15, 1955. Presented to the American Mathematical Society, April 24, 1954. The author wishes to express appreciation to Professor F. B. Jones for having stimulated this research in classes at The University of North Carolina.

[1] A topological space S is said to be a semi-metric topological space provided there is a distance function d defined for S such that (1) if each of the letters x and y denotes a point of S, then $d(x, y)=d(y, x)$ denotes a non-negative number, (2) $d(x, y)=0$ if and only if $x=y$, and (3) the topology of S is invariant with respect to the distance function d, that is, if p is a limit point of a subset M of S, then p is a distance limit point of M and conversely. As usual, S is said to be regular provided that if R is an open set containing a point p of S, then there exists an open set D such that $R\supset \bar{D}\supset p$. A topological space (T_1) is defined as in [9].