# DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL $n$-SPACE

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**Introduction.** This paper is a generalization to $n$ dimensions of the classification of the differentiable points in the conformal plane [2], and in conformal 3-space [3]. In the present paper, this classification depends on the intersection and support properties of certain families of tangent $(n-1)$-spheres, and on the nature of the osculating $m$-spheres at such a point $(m=1, 2, \cdots, n-1)$.

The discussion is also related to the classification [4] of the differentiable points of arcs in projective $(n+1)$-space, since conformal $n$-space can be represented on the surface of an $n$-sphere in projective $(n+1)$-space.

**1. Pencils of $m$-spheres.** $p, t, P, P_1, \cdots,$ will denote points of conformal $n$-space and $S^{(m)}$ will denote an $m$-sphere. When there is no ambiguity, the superscript $(n-1)$ will be omitted in the case of $S^{(n-1)}$; thus an $(n-1)$-sphere $S^{(n-1)}$ will usually be denoted by $S$ alone. Such an $(n-1)$-sphere $S$ decomposes the $n$-space into two open regions, its *interior* $\underline{S}$, and its *exterior* $\overline{S}$. If $P \not\subset S$, the interior of $S$ may be defined as the set of all points which do not lie on $S$ and which are not separated from $P$ by $S$; the exterior of $S$ is then defined as the set of all points which are separated from $P$ by $S$. An $m$-sphere through an $(m-1)$-sphere $S^{(m-1)}$ and a point $P \not\subset S^{(m-1)}$ will be denoted by $S^{(m)}[P; S^{(m-1)}]$. The $m$-sphere through $(m+2)$-points $P_0, P_1, \cdots, P_{m+1}$, not all lying on the same $(m-1)$-sphere, will occasionally be denoted by $S^{(m)}(P_0, P_1, \cdots, P_{m+1})$. Such a set of points is said to be *independent*. Most of the following discussion will involve the use of pencils $\pi^{(m)}$ of $m$-spheres determined by certain incidence and tangency conditions. An $(m-1)$-sphere which is common to all the $m$-spheres of a pencil $\pi^{(m)}$ is called *fundamental* $(m-1)$-*sphere* of $\pi^{(m)}$. In the pencil $\pi^{(m)}$ through a fundamental $(m-1)$-sphere $S^{(m-1)}$ there is one and only one $m$-sphere $S^{(m)}(P, \pi^{(m)})$ of $\pi^{(m)}$ through each point $P$ which does not lie on $S^{(m-1)}$. Similarly, in the pencil $\pi^{(m)}$ of all the $m$-spheres which touch a given $m$-sphere at a given point $Q$, there is one and only one $m$-sphere $S^{(m)}(P, \pi^{(m)})$ through each point $P \neq Q$. The *fundamental point* $Q$ is regarded as a *point* $m$-*sphere* belonging to $\pi^{(m)}$.

Received May 18, 1955. This paper was prepared while the author held a fellowship at the Summer Research Institute of the Canadian Mathematical Congress.