ON THE SPECTRA OF LINKED OPERATORS

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1. Introduction. Let X, Y be complex linear spaces, and Z a non-void complex linear space contained in both X and Y. Let X be a Banach space X_1, Y a Banach space Y_2 under the norms n_1, n_2 respectively. Let Z be a Banach space Z_N under the norm N defined by $N(z)=\max[n_1(z), n_2(z)]$. (This is equivalent to saying that if $\{z_n\}$ is any sequence with $z_n \in Z$, such that $z_n \to x$ in the topology of X_1 and $z_n \to y$ in the topology of Y_2, then $x=y \in Z$. Our particular method of stating this here will be useful for later purposes.) With the usual uniform norms let T_1, T_2 be bounded distributive operators on X_1, Y_2 respectively, such that $T_1 z = T_2 z \in Z$ when $z \in Z$. Operators satisfying these conditions will be said to be "linked". If, in addition, it is assumed that Z is dense in X_1, T_1 and T_2 will be said to be "linked densely relative to X_1".

We are interested in relationships between the spectra of linked operators. That there are linked, and densely linked operators with different spectra will be shown in § 3. The main result of this paper is the demonstration that, if T_1 and T_2 are linked densely relative to X_1, under certain circumstances any component of the spectrum of T_1 has a non-void intersection with the spectrum of T_2. Sufficient conditions are that if λ belongs to the intersection of the resolvent sets of T_1 and T_2 and $z \in Z$, then $(\lambda I - T_1)^{-1} z = (\lambda I - T_2)^{-1} z \in Z$. With this result we obtain some interesting consequences in the special case where the Banach spaces considered are the sequence spaces l_p.

2. Preliminary definitions and notation. Supposing X to be a complex linear space such that under a norm n_a, ($x \in X$, $n_a(x) = \|x\|_a$), X becomes a complex Banach space X_a, we let $[X_a]$ denote the set of all operators T that are bounded under the induced norm

$$\|T\|_a = \sup \|Tx\|_a \text{ (for all } x \in X_a,\ \|x\|_a = 1).$$

Such a T will be denoted by T_a when considered as an element of the algebra $[X_a]$. If $T_a \in [X_a]$ we classify all complex numbers into two sets:

(1) The resolvent set $\rho(T_a)$, consisting of all λ such that $\lambda I - T_a$ defines a one-to-one correspondence of X_a onto X_a.

(2) The spectrum $\sigma(T_a)$, consisting of all λ not in $\rho(T_a)$.

The spectrum is divided into three parts:

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