MEAN VALUES OF HARMONIC FUNCTIONS ON HOMOTHETIC CURVES

JOHN W. GREEN

1. Introduction. It is well known indeed that if $U(P)$ is harmonic in the plane the mean value of U taken over the perimeter or the area of a circle with center P_0 equals $U(P_0)$. A related result is that of Ásgeirsson [1] which states that mean of U over the area of any one of a family of confocal ellipses equals that over any other. The same is true for the means over the perimeters, provided the means are weighted by integrating with respect to the anomaly angle instead of arc length.

It would be interesting to know if there are any other simple families of curves over which the perimeter or area average is constant. The simplest families to try are homothetic families, and in the following we show that under suitable regularity assumptions, there are none of these except circles.

2. Perimeter means. Let C be a closed simple rectifiable curve containing O in its interior. We suppose that C is smooth enough that its Green's function $g(P)$ with pole at O is continuously differentiable on C, as will be the case if C has a continously turning tangent line. By C_λ we mean the curve obtained from C through the homothetic transformation $x'=\lambda x$, $y'=\lambda y$. If we expect that for $\lambda<1$, C_λ is inside C, we should assume that C is star-shaped about O, although this is not essential to what follows. A positive continuous weight function $w(P)$ is given, and we suppose that the mean of U with weight w over all C_λ is constant, provided U is harmonic inside and on C.

We immediately note two things. In the first place, letting λ tend to zero, we see that the mean value of U over C_λ must be $U(O)$. In the second place, from obvious continuity considerations, we see that U need only be harmonic inside C and continuous on C for the mean over C_λ to be constant. Since U may be given arbitrary continuous values on C and determined inside so as to be harmonic, it follows that

$$U(O)=\frac{1}{W}\int_C U(P)w(P)ds$$

for every continuous U, where $W=\int_C w(P)ds$. But also

Received April 22, 1955.