EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A NON-SELF-ADJOINT DIFFERENTIAL EQUATION

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1. Introduction. In solving certain characteristic boundary-value problems by the method of separation of variables [2], the problem arose of expanding an arbitrary function $f(x)$ in terms of the eigenfunctions of the equation $(A+\lambda B)u=0$, where A is a second-order and B a first-order differential operator. In this paper we consider a special case of this problem, namely the following:
Expand a function $f(x)$ in terms of the eigenfunctions of the equation

$$u''+q(x)u+\lambda(p(x)u-u')=0\,, \tag{1.1}$$

where $u(0)=u(1)=0$. There has been a long series of investigations concerned with the corresponding self-adjoint problem for the equation $(A-\lambda)u=0$, which often occurs in connection with the boundary-value problems of mathematical physics. However, the problem we are concerned with here does not seem to have been considered previously. F. Browder [1] has considered the eigenfunctions of $A+\lambda B$ where A and B are general partial differential operators, but he has always assumed that B is positive definite. We shall show that the lack of definiteness in B gives rise to peculiar results in the expansion theorem. R. E. Langer [3] has considered the expansion theorem for the following equation, which is similar to (1.1)[1].

$$u''+\{p_{11}\lambda+p_{10}\}u'+\{p_{22}\lambda^2+p_{21}\lambda+p_{20}\}u=0\,.$$

This equation of course reduces to (1.1) if we put

$$p_{10}=p_{22}=0\,,\qquad p_{11}=-1\,,\qquad p_{21}=p,\qquad p_{20}=q.$$

However, Langer in his paper made the assumption that the roots of $r^2+p_{11}r+p_{22}=0$ were distinct and nonvanishing. For (1.1), it is clear that $r=0$, $r=+1$, and hence Langer's conditions do not apply. In fact, the results we shall obtain are strikingly different from those of Langer.

Since the operator B is not self-adjoint, we must also consider the adjoint of (1.1), namely

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1) A detailed treatment of this expansion problem and related questions has been given by Titchmarsh [4].