# ON A CLASS OF POLYNOMIALS ORTHOGONAL OVER A DENUMERABLE SET

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**1. Introduction and Summary.** This study proceeds from a theorem of Favard that states that polynomial sets obeying a certain type of three-term recursion formula are, with respect to some weight function, orthogonal over some set on the real axis. For a rather wide subclass of these polynomial sets, it is shown that the orthogonality set consists of a discrete but infinite set of points of the real axis. The orthogonality set and the weight function are given by certain entire functions that can be constructed from the polynomials by a limiting process. A set of modified Lommel polynomials is given as an explicit example of the general theory. They are orthogonal over the reciprocals of the roots of certain Bessel functions.

**2. General theory and existence of orthogonality.** In 1935, Favard [2] sketched the proof of the following theorem:

*If there is a sequence of real polynomials $\{\varphi_n(x)\}$ possessing a recursion relation of the form*

$$\varphi_{n+1}(x)=(x-a_n)\varphi_n(x)-\lambda_n\varphi_{n-1}(x) , \qquad (n\geq 0) , \tag{1}$$

*with*

$$\varphi_0(x)=1 , \quad \varphi_1(x)=x-a_0 , \quad \varphi_{-1}(x)=0 \tag{2}$$

*and*

$$\lambda_n>0 , \tag{3}$$

*then the polynomials are orthogonal.*

This powerful theorem seems to have escaped attention; it suggests that whenever a relation of the form (1) exists, a weight function and orthogonality domain could probably be found by detailed study. We want to accomplish this for a wide class of polynomials for which the orthogonality range comes out to be a denumerable set of points.

The class of polynomials mentioned is obtained by imposing upon (2) and (3) the restrictions that

$$a_n=0 \qquad (n\geq 0) \tag{4}$$

and that the limit $B$ exists where

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