INVARIANT FUNCTIONALS

PAUL CIVIN AND BERTRAM YOOD

1. Introduction. Let E be a normed linear space and G a solvable group of bounded linear operators on E. If there exists a non-trivial bounded linear functional invariant under G then there exists $x_0 \in E$ such that $\inf \|T(x_0)\| > 0$, $T \in G_1$, the convex envelope of G. Assume that such an x_0 exists. If G is bounded then there exists an invariant functional [7]. If G is unbounded, however, such a functional may or may not exist.

For simplicity we discuss here the abelian case. In a previous work [7] it was shown that the invariant functional exists if there is a constant $K > 0$ such that to each $U \in G_1$ there corresponds $V \in G_1$ where $\|V\| \leq K$ and $\|VU\| \leq K$. A consequence of this condition is that for each $x \in E$

$$\inf_{\substack{\|T\| \leqq K \\ T \in G_1}} \|T(x)\| \leq K \inf_{T \in G_1} \|T(x)\| . \tag{1}$$

Now call an element y *stable* if (1) holds for some $K = K(y)$ for all x of the form $U(y)$, $U \in G_1$. We show here that the invariant functional exists if E is complete and if there exists an open set S in E such that for all $x \in S$, $T \in G$, x and $T(x) - x$ are stable. An analogous result is shown to hold if G is solvable.

The problem of the existence and extension of functionals invariant under solvable groups of operators has been considered by Agnew and Morse and by Klee (see [3] for references). These authors use for E any real linear space while we take E to be a Banach space in order to utilize category arguments.

2. Notations. Let E be a Banach space and $\mathfrak{E}(E)$ be the set of all bounded operators on E. Let H be a (multiplicative) semi-group in $\mathfrak{E}(E)$. By H_1 we mean the convex envelope of H (the smallest convex subset of $\mathfrak{E}(E)$ which contains H). As in [7] we adopt the following notation. By $B(H)$ we mean the linear manifold generated by elements of the form $T(x) - x$, $x \in E$, $T \in H$. By $Z(H)$ we mean $\{x \in E \mid \inf \|T(x)\| = 0, T \in H\}$.

We introduce the following notation. An element $x \in E$ is *stable* with respect to H if there exist positive numbers K, L such that

$$\inf_{\substack{\|T\| \leq K \\ T \in H}} \|T(y)\| \leq L \inf_{T \in H} \|T(y)\|$$

for all y of the form $U(x)$, $U \in H$.

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