CORRESPONDING RESIDUE SYSTEMS IN ALGEBRAIC NUMBER FIELDS

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In this paper we shall consider integral ideals in finite algebraic extensions of the field R of rational numbers. Algebraic number fields will be denoted by $\mathfrak{F}$ with subscripts or superscripts, ideals by German letters, algebraic numbers by lower case Greek letters, and numbers of the rational field R by lower case Latin letters.

Two ideals in the same field are equal if and only if they contain the same numbers.

If $\mathfrak{a}_1$ is an ideal in a field $\mathfrak{F}_1$ and $\mathfrak{a}_2$ is an ideal in a field $\mathfrak{F}_2$, then we shall write $\mathfrak{a}_1=\mathfrak{a}_2$ provided $\mathfrak{a}_1$ and $\mathfrak{a}_2$ generate the same ideal in some field containing all the numbers of $\mathfrak{F}_1$ and of $\mathfrak{F}_2$ (see [**1**, § 37]). Two such ideals may therefore be denoted by the same symbol and we shall speak of an ideal $\mathfrak{a}$ without regard to a particular field. An ideal $\mathfrak{a}$ is said to be contained in a field $\mathfrak{F}$ if it may be generated by numbers in $\mathfrak{F}$, that is to say, if it has a basis in $\mathfrak{F}$.

Let $\mathfrak{a}$ be an ideal contained in the fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$. We say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have *corresponding residue systems modulo* $\mathfrak{a}$ if for every integer α_1 of $\mathfrak{F}_1$ there exists an integer α_2 of $\mathfrak{F}_2$ such that $\alpha_1\equiv\alpha_2$ (mod $\mathfrak{a}$), and for every integer α_2 of $\mathfrak{F}_2$ there exists an integer α_1 of $\mathfrak{F}_1$ such that $\alpha_1\equiv\alpha_2$ (mod $\mathfrak{a}$).

The problem considered in this paper is the following one: if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two fields containing an ideal $\mathfrak{a}$, under what conditions will $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have corresponding residue systems mod $\mathfrak{a}$. We shall show that this problem reduces to that in which the ideal $\mathfrak{a}$ is a power of a prime ideal and a necessary and sufficient condition for $\mathfrak{F}_1$ and $\mathfrak{F}_2$ to have corresponding residue systems mod $\mathfrak{a}$ is derived in case that $\mathfrak{a}$ is a prime ideal. A necessary (but not sufficient) condition is derived in case $\mathfrak{a}$ is a power of a prime ideal and $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are normal over $\mathfrak{F}_1\cap\mathfrak{F}_2$. A special case in which the fields are of the type $\mathfrak{F}(\sqrt[q]{\mu})$ is considered. These fields are of interest in themselves and in view of Corollary 7.1 seem to have a direct connection with the general problem.

THEOREM 1. *Let $\mathfrak{a}$ be an ideal in the number fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$ and suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have corresponding residue systems mod $\mathfrak{a}$. Then $\mathfrak{a}$ has the same prime ideal decomposition in $\mathfrak{F}_1$ and in $\mathfrak{F}_2$.*

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