A CLASS OF MEASURE PRESERVING TRANSFORMATIONS

CHARLES STANDISH

In this paper we shall consider the following class of transformations of the unit interval onto itself. Let π be a permutation of the positive integers, that is, a one-to-one mapping of the positive integers onto themselves. Let t $(0 \leq t \leq 1)$ be represented in its dyadic expansion:

$$t=\sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{2^k}, \qquad \varepsilon_k=0 \text{ or } 1 .$$

Then we define

$$T_\pi(t)=\sum_{k=1}^{\infty} \frac{\varepsilon_{\pi(k)}(t)}{2^k} .$$

$T_\pi(t)$ "shuffles" the digits in the dyadic expansion of t.

Our motivation in considering these transformations lies in the fact that they form a nontrivial class of measurable transformations with a simple intuitive interpretation and may be utilized to illustrate several of the concepts of ergodic theory.

1. Measurability and ergodicity considerations.

THEOREM 1.1. *For every choice of π, T_π is a measure preserving transformation.*

Proof, Let X_i $(i=1, 2\cdots)$ be the space consisting of the two real numbers 0 and 1 endowed with a measure m defined by $m(0)=1/2$ $m(1)=1/2$. Consider the product space $X=\prod_{i=1}^{\infty} X_i$ (where we omit those products for which all but a finite number of factors$=1$) and define the measure of a "rectangle" $\prod_{i=1}^{\infty} E_i$, $E_i \subset X_i$ by $\mu(\prod_{i=1}^{\infty} E_i)=\prod_{i=1}^{\infty} m(E_i)$ then it can be shown [1, p. 159] that the above measure is capable of extension to a measure on a σ algebra of subsets containing the rectangles in such a fashion that the mapping

$$\varphi:\ X \to [0, 1]$$

defined by

Received February 14, 1955, and in revised form August 4, 1955. This is a portion of a doctoral thesis written at Cornell University under the direction of Professor Mark Kac. The author wishes to thank Professor Kac for his guidance and assistance.