SIMPLE FAMILIES OF LINES

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1. Introduction. Planar families of lines are studied by P. C. Hammer and the author in [2], and families of lines in the plane and in ordinary space by the author in [6]. Families of lines in vector spaces E_3 and E_n are mentioned in connection with convex bodies in [1]. The present paper gives a classification of simple types of families in $(n+1)$ dimensional real vector space E_{n+1}. Theorems are obtained on relations between the type of the family F, and the properties which F may possess, of containing exactly one line in every direction, and of simply or multiply covering the points of E_{n+1}.

2. Notation and definitions. With respect to an n dimensional vector subspace E_n of $(n+1)$ dimensional real vector space E_{n+1} a line L in E_{n+1} will be called *horizontal* if it is parallel to E_n. Any family F of non-horizontal lines in E_{n+1}, for which there is a hyperplane H parallel to E_n such that each point of H is covered exactly once by F, determines a single valued function $y=f(x)$ on H to any parallel hyperplane K: x, y are the points in which the line L of F which covers x intersects H, K. Corresponding to any basis in E_n, and choice of origins in H, K, the function $f(x)$ will be represented by real valued functions $y_i=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$. (For definiteness, let E_{n+1} be Euclidean, and choose the origins in H, K to be their points of intersection with the line through the common origin of E_{n+1}, E_n, which is orthogonal to E_n.)

A family F will be said to be *composed* of two lower dimensional *associated* families, F_p and F_{n-p}, if there is a choice of basis such that the n real functions have the form $y_i=f_i(x_1, \cdots, x_p)$, $i=1, \cdots, p$; $y_j=f_j(x_{p+1}, \cdots, x_n)$, $j=p+1, \cdots, n$. (The dimension of an associated family of course is one greater than the subscript; thus for example a three dimensional family may be composed of two associated two dimensional families.)

A family F is *primary* if it contains exactly one line in every non-horizontal direction, *representative* if it contains exactly one line in every direction. We say that a family F of lines is *simple* if every point of E_{n+1} is covered exactly once by the family; *outwardly simple* if every point exterior to some sphere S_n has the same property in relation to the family. If the distances from the origin of the lines of an outwardly simple family are bounded, then for a sufficiently large sphere S_n, if P, $g(P)$ are the points in which the line L of F covering P pierces S_n,

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