ON CERTAIN CHARACTER MATRICES

D. H. LEHMER

For only a very limited class of matrices M is it possible to give explicit formulas for the determinant, characteristic roots and inverse of M as well as the general element of M^k. Nontrivial instances of such sets of matrices are useful as examples in testing the correctness and efficacy of various matrix computing routines especially when the elements are small integers or simple rational numbers. The purpose of this paper is to indicate two new classes of such matrices which arise from the theory of exponential sums and have as general elements simple functions involving real nonprinciple characters or Legendre symbols.

The same method of determining characteristic roots is used for both types of matrices. It depends on the fact that the roots of a polynomial are determined by the sums of their like powers. That is, if S_k denotes the sum of the kth powers of the roots of a polynomial of degree n and if complex numbers $\rho_1, \rho_2, \cdots, \rho_n$ are exhibited for which

$$\sum_{i=1}^{n} \rho_i^k = S_k \qquad (k=1, \cdots, n),$$

then the ρ_i are the roots of the polynomial.

All matrices M are square and of order $p-1$ where p is an odd prime. Their elements involve Legendre's symbol $\chi(n)$ defined by

$$\chi(n)=\left(\frac{n}{p}\right)=\begin{cases} 0 \text{ if } p \text{ divides } n \\ -1 \text{ if the congruence } x^2\equiv n \pmod{p} \text{ is impossible} \\ +1 \text{ otherwise} \end{cases}$$

Thus for $p=7$

$$\chi(0)=0 \qquad \chi(1)=1 \qquad \chi(2)=1 \qquad \chi(3)=-1$$

$$\chi(4)=1 \qquad \chi(5)=-1 \qquad \chi(6)=-1 .$$

Besides the simple properties

$$\chi(i)\chi(j)=\chi(ij)$$

$$\chi(i+p)=\chi(i)$$

Received August 26, 1955. A part of the results of this paper were obtained in 1952 under a contract between the Office of Naval Research and the National Bureau of Standards.