# CONVERGENCE TOPOLOGIES FOR MEASURES AND THE EXISTENCE OF TRANSITION PROBABILITIES

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**1. Introduction.** A recent approach to measure theory is the introduction of measures as functionals on spaces of continuous functions [2]. To the probabilist, however, the measures are of primary concern, with the functions occurring as integrands playing a secondary role as random variables. We are thus motivated to reverse the modern procedure. We shall introduce various topologies into spaces of measures and shall in each case investigate the dual space consisting of all continuous linear functionals on the measures. From this point of view the continuous functions form only one of many possible dual spaces to a space of measures.

The study of the dual spaces yields a necessary and sufficient continuity condition for the existence of transition probabilities in a stochastic semigroup, thus solving a problem posed by W. Feller.

We introduce topologies through the convergence of nets [5], an elegant device for analysis. The spaces of measures considered are vector spaces and usually vector lattices [1]. We admit any topology for which the vector operations are continuous, but do not require that the lattice operations be continuous.

Let $\mathfrak{A}$ be a Boolean algebra of sets in an abstract space $X$. Wherever $\mathfrak{A}$ is required to be a $\sigma$-algebra, it shall be denoted by $\mathfrak{A}_\sigma$. A *partition* $\rho$ of $X$ is a finite collection $\{E_k\}$ of sets in $\mathfrak{A}$ which form a disjoint covering of $X$. The partitions of $X$ form a lattice [1] if we define $\rho < \rho'$ whenever $\rho'$ is a refinement of $\rho$. In this way the partitions $\rho$ will be used extensively as directed indices for nets.

For each $x$ in $X$ define the *unit point mass* $H_x$ by

$$H_x(E) = \begin{cases} 0 & \text{if } x \text{ is not in } E. \\ 1 & \text{if } x \text{ is in } E. \end{cases} \tag{1.2}$$

Such $H_x$ will belong to all spaces of measures considered below. A *discrete measure* is any finite linear combination of point masses. We shall use the symbol $E$ to denote the characteristic function of a set $E$, since the context will serve to distinguish between the set and its characteristic function. Thus,

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