NEIGHBOR RELATIONS ON THE CONVEX OF CYCLIC PERMUTATIONS

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1. Introduction and summary. Two vertices of a polyhedron are called neighbors of order k when they have a face of dimension k, and none of lower dimension, in common. $K(P)$ denotes the maximum value of k for a given polyhedron P. For the convex hull (polyhedron) P_n of all permutations of n elements (represented by square matrices of order n and interpreted as points in n^2-space) it was shown [1 and 2] that $K(P)=[n/2]$ (that is, the largest integer not exceeding $n/2$), which is rather small as compared with $\dim P_n=(n-1)^2$. For the convex hull Q_n of all cyclic permutations of n elements that leave no element fixed, *H.* Kuhn performed computations showing that any two vertices of Q_5 but not any two vertices of Q_6 are neighbors of order 1, which means that $K(Q_5)=1$ and $K(Q_6)>1$. The present note, dealing with general n, proves, for $n\geqq 8$:

$$K(Q_n)=K(P_n)-1=\frac{n}{2}-1 \quad \text{if} \quad n=4m+2 \tag{1}$$

$$K(Q_n)=K(P_n)=\left[\frac{n}{2}\right] \quad \text{if} \quad n\neq 4m+2 \tag{2}$$

For $n=1, 2, \cdots 6, 7$, $K(Q_n)=0, 0, 1, 1, 1, 2, 2$ respectively.

2. A permutation p of n numbered elements is customarily represented by a matrix (p_{ij}), where

$$p_{ij}=\begin{cases} 1 & \text{when } p \text{ sends } i \text{ into } j \\ 0 & \text{otherwise.} \end{cases}$$

To the product of permutations then corresponds the product of the associated matrices under ordinary matrix multiplication, and therefore the same symbol will be used for a permutation and its matrix.

The following facts from [1] and [2] regarding neighbor relations on P_n will be used in the sequal:

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