ON LIE ALGEBRAS OF ALGEBRAIC LINEAR TRANSFORMATIONS

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1. Introduction. This paper is concerned with the structure of locally finite Lie algebras of algebraic linear transformations in an infinite dimensional space. The main results include generalizations of the Cartan-Jacobson theorem [7] on completely reducible Lie algebras of linear transformations, Lie's theorem [3] on irreducible representations of solvable Lie algebras, and Malcev's theorem [14] on the structure of splittable algebras. The methods of proof are essentially the known ones, except than in each case suitable reductions to finite dimensional situations must be found.

We obtain the following two criteria in order that a Lie algebra $\mathfrak{L}$ of algebraic linear transformations be commutative:

(A) $\mathfrak{L}$ is solvable, and the enveloping associative algebra $\mathfrak{A}$ of $\mathfrak{L}$ is semi-simple in the sense of Jacobson [9];

(B) $\mathfrak{L}$ is solvable, of countable dimension over an algebraically closed field, and the linear transformations in $\mathfrak{L}$ are semi-simple. An example shows that in neither case need all the row-finite matrices corresponding to the elements of $\mathfrak{L}$, relative to a fixed basis of the vector space, be diagonal, contrary to the situation which prevails in the finite dimensional case.

We summarize briefly the contents of the paper. It is proved first, in greater generality than is needed for this paper, that a locally algebraic linear transformation A can be expressed uniquely as a sum $A=S+N$ of a semi-simple locally algebraic linear transformation S and a locally nilpotent transformation N, both of which commute which A, and can be approximated by polynomials in A. In § 3 it is proved that a locally finite Lie algebra $\mathfrak{L}$ of algebraic linear transformations whose enveloping associative algebra is semi-simple, is a direct sum of an ideal $\mathfrak{L}_1$ containing $[\mathfrak{L}, \mathfrak{L}]$, and the center $\mathfrak{C}$; moreover $\mathfrak{L}_1$ contains no nonzero solvable ideals, and every element of $\mathfrak{C}$ is semi-simple. An example is given which indicates that $[\mathfrak{L}, \mathfrak{L}]$ may be properly contained in $\mathfrak{L}_1$. In § 4, Theorem (A) is proved, together with the fact that any solvable Lie algebra of algebraic linear transformations is locally finite. The final section contains a discussion of linearly splittable algebras, the generalization of Malcev's theorem, Theorem (B), and an example which shows that the countability hypotheses which are introduced in the section cannot be removed.

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