EXTENSION OF UNIFORMLY CONTINUOUS TRANSFORMATIONS AND HYPERCONVEX METRIC SPACES

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Introduction. The results of the present paper combine the research done by the first author mainly in 1929–1930 (which was never published) and the results of the thesis presented by the second author at the University of Kansas, 1955.

The principal topic of the first two sections is the following: can a uniformly continuous transformation T of a metric space $\mathscr{D}$ into a metric space $\mathscr{F}$ be extended with conservation of modulus of continuity to any larger metric space $\mathscr{E}$ containing $\mathscr{D}$ metrically so that the range is still contained in $\mathscr{F}$? In § 1 we show (Theorem 2) that for the possibility of such unlimited extension of T it is necessary that the minimal modulus of continuity of T satisfy a condition which is proved in Theorem 1 to be necessary and sufficient for the existence of a subadditive modulus of continuity for T. In § 2 the transformations T are restricted to be those with a subadditive modulus of continuity. The main result of this section is that a necessary and sufficient condition that there exist an unlimited extension of any transformation T into a space $\mathscr{F}$ with conservation of a subadditive modulus of continuity $\delta(\varepsilon)$ is that $\mathscr{F}$ be hyperconvex[1] (see Definition 1 of § 2). The $\mathfrak{m}$-hyperconvexity is introduced for any cardinal $\mathfrak{m} \geqq 3$, which is a weaker property than hyperconvexity.

In § 3 the properties of hyperconvex (or $\mathfrak{m}$-hyperconvex) spaces and subsets of metric spaces are investigated. As a useful tool the notion of almost $\mathfrak{m}$-hyperconvexity is introduced; it is slightly weakar than $\mathfrak{m}$-hyperconvexity. The main results of this section are the following: $\mathfrak{m}$-hyperconvexity implies completeness for $\mathfrak{m} > \aleph_0$ (Theorem 1′); almost $\mathfrak{m}$-hyperconvexity and completeness imply $\mathfrak{m}$-hyperconvexity for $\mathfrak{m} \geqq \aleph_0$ (Theorem 4). In any complete metric space the class of all $\mathfrak{m}$-hyperconvex subsets is considered as a subset of the class of all closed subsets provided with the well known metric introduced by Hausdorff. It is proved that $\mathfrak{m}$-hyperconvex subsets form a closed set in the class of all closed subsets (Theorem 5). The topological properties of hyperconvex spaces are then investigated. It is proved that every hyperconvex space is a generalized absolute retract.

In § 4 the hyperconvex Banach spaces are considered and a direct

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[1] This suggestive term "hyperconvex" was proposed to the authors by A. H. Kruse.