NOTE ON A THEOREM OF HADWIGER

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Throughout this paper, H denotes a Hilbert space over the real or complex numbers and (x, y) denotes the inner product of the vectors x, y of H. The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if $\{u_\alpha\}$ is an orthonormal basis of H, then the Parseval relation

$$(1) \qquad (x, y)=\Sigma(x, u_\alpha)(u_\alpha, y)$$

is valid for each pair of vectors x, y of H. It is easy to see that (1) is also valid if $\{u_\alpha\}$ is the projection of an orthonormal basis $\{w_\alpha\}$ and if we restrict x and y to the range of the projection. Indeed, if E is the projection, so that $w_\alpha E=u_\alpha$ for each α, then

$$(x, y)=\Sigma(x, w_\alpha)(w_\alpha, y)=\Sigma(xE, w_\alpha)(w_\alpha, yE)=\Sigma(x, w_\alpha E)(w_\alpha E, y)$$
$$=\Sigma(x, u_\alpha)(u_\alpha, y) .$$

The theorem referred to in the title deals with this result and also with the converse question:

THEOREM 1. *If the Parseval relation* (1) *is valid for each pair of vectors x and y of H, then the set $\{u_\alpha\}$ is the projection of an orthonormal basis of a superspace K of H.*

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

Proof of Theorem 1. We choose as K coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set $\{u_\alpha\}$. We see from (1), with $x=u_\beta$, $y=u_\gamma$, that the matrix $U=((u_\alpha, u_\beta))$ is idempotent. Since U is also Hermitian, it may be interpreted as a projection acting on K. We now imbed H in K by making correspond to x in H the (row) coordinate vector $x'=\{(x, u_\alpha)\}$ in K. In particular, to the vector u_β there corresponds the βth row of U which is manifestly the image, under the projection U, of the βth coordinate basis vector. Finally, if $x'=\{(x, u_\alpha)\}$ and $y'=\{(y, u_\alpha)\}$, then $(x', y')=\Sigma(x, u_\alpha)\overline{(y, u_\alpha)}=\Sigma(x, u_\alpha)(u_\alpha, y)$ $=(x, y)$; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

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