ON THE TWO-ADIC DENSITY OF REPRESENTATIONS BY QUADRATIC FORMS

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1. Introduction. The problem of determining $A_q(S, T)$, the number of solutions of $X'SX \equiv T \pmod q$, where $S^{(m)}$ and $T^{(n)}$ are symmetric integral matrices, has been considered by C. L. Siegel [2, pp. 539–547]. He obtained explicit formulas for $A_q(S, T)$ when $q=p^a$, where p is a prime not dividing $2|S||T|$. We wish to determine both $A_2(S, T)$ and $A_8(S, T)$ when $|S||T|$ is odd. Siegel has shown that the calculation of $A_8(S, T)$, for $|S||T|$ odd, is sufficient to give results when the modulus is replaced by a higher power of 2. Moreover, his work for composite moduli does not exclude a power of 2 as a factor.

We shall follow the pattern of Siegel's work, modifying it by the use of canonical forms established by B. W. Jones [1, pp. 715–727] and Gordon Pall for symmetric matrices in G_2, the ring of 2-adic integers. (Clearly, $A_q(S, T)$ depends only on the classes of S and T in G_q, the ring of q-adic integers). We shall calculate $A_2(S, T)$ combinatorially and $A_8(S, T)$ by the use of exponential sums.

2. Recursion formula. For convenience, we state here the following theorem of Jones:

Every quadratic form with matrix in G_2 and with unit determinant, D, is equivalent to one of the following:

(a) $$x_1^2+x_2^2+\cdots+ax_{r-2}^2+bx_{r-1}^2+cx_r^2 ,$$

where a, b, c take one of the following sets of values:

$$(1, 1, 1) \text{ or } (1, 3, 3) \text{ for } D\equiv 1 \pmod 8 ,$$
$$(1, 1, 5) \text{ or } (1, 3, 7) \text{ for } D\equiv 5 \pmod 8 ,$$
$$(1, 1, 3) \text{ or } (3, 3, 3) \text{ for } D\equiv 3 \pmod 8 ,$$
$$(1, 1, 7) \text{ or } (3, 3, 7) \text{ for } D\equiv 7 \pmod 8 ,$$

while if $r=2$, b and c take one of the following sets of values:

$$(1, 1) \text{ or } (3, 3) \text{ for } D\equiv 1 \pmod 8 ,$$
$$(1, 5) \text{ or } (3, 7) \text{ for } D\equiv 5 \pmod 8 ,$$
$$(1, 3) \qquad \text{ for } D\equiv 3 \pmod 8 ,$$
$$(1, 7) \qquad \text{ for } D\equiv 7 \pmod 8 .$$

(b) A sum of binary forms of the two types: $f=2x_1^2+2x_1x_2+2x_2^2$,

Received October 3, 1955. The author wishes to thank Professor Irving Reiner for helpful suggestions during the preparation of this paper.