A FUNCTIONAL INDEPENDENCE THEOREM FOR SQUARE MATRICES

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The purpose of this paper is to prove the following independence theorem:

If A is a square matrix of order n, the Jacobian of the n traces of $A, A^2, \cdots, A^n$ with respect to each set of n distinct elements of A, at least one of which is a diagonal element, is never identically zero in the n^2 elements of A.

The problem arose originally in connection with a certain system of differential equations of the second order [1]. This led to the investigation of the properties of a class of determinants which are generalizations of the classical determinant of Vandermonde [2]. The latter half of [2] includes a proof of the independence theorem as given by Perron who used mathematical induction. We now give the proof first devised by the authors in 1940. It is interesting for two reasons; first, new results in the algebra of matrices are brought to light and second, matrices are constructed for which the n traces are independent.

1. Notations and terminology. Let $A=(a_{ij})$ be a square matrix of order n whose elements are independent indeterminates over an arbitrary field. Let $a_{ij}^{(m)}$ stand for the element in the ith row and jth column of the mth power of A. The determinant

$$V_s^r(A)=\begin{vmatrix} \delta_{r_1s_1} & \delta_{r_2s_2} & \cdots & \delta_{r_ns_n} \\ a_{r_1s_1} & a_{r_2s_2} & \cdots & a_{r_ns_n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r_1s_1}^{(n-1)} & a_{r_2s_2}^{(n-1)} & \cdots & a_{r_ns_n}^{(n-1)} \end{vmatrix}$$

where $r_1, \cdots, r_n$ and $s_1, \cdots, s_n$ are arbitrary integers in the range 1 to n, equal or unequal, and δ_{ij} is the Kronecker delta, is called a generalized determinant of Vandermonde. It reduces to the classical determinant of Vandermonde if A is a diagonal matrix.

Consider any set S of n distinct elements $a_{s_1r_1}, a_{s_2r_2}, \cdots, a_{s_nr_n}$ of the matrix A. Also consider the set T of n traces $t_1, t_2, \cdots, t_n$ of A, $A^2, \cdots, A^n$. Let us represent by $\partial T/\partial S$ the Jacobian of the set T with

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