COMPLETELY MONOTONIC FUNCTIONS ON CONES

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1. Introduction. A function $f(x)$, $0 \leq x < \infty$, is said to be completely monotonic on $0 \leq x < \infty$ if $(-1)^n f^{(n)}(x) \geq 0$ for $0 < x < \infty$ and $f(0) = f(0+)$. A similar and equivalent definition involving differences is available. A fundamental theorem regarding such functions, proved (independently) by Hausdorff, Bernstein and Widder, states that they are the class of Laplace-Stieltjes transforms of bounded monotone functions. Several of the many known proofs are given in Widder [3], which also gives references for other proofs. The corresponding theorem for two dimensions has been proved by Schoenberg [2]. It is not difficult to construct a proof for n-dimensions along the lines of the original proof of Hausdorff and in the process establish the equivalence of the corresponding derivative and difference criteria.

In this note we wish to introduce a class of functions, defined on n-dimensional polyhedral cones with vertex at the origin, which we call completely monotonic (A), and, in analogy with the theorem of Hausdorff-Bernstein-Widder, show that they are the Laplace-Stieltjes transforms of bounded monotone functions on the "conjugate space" $t = (t_1, \cdots, t_n)$ with $\sum_{i=1}^{n} x_i t_i \geq 0$. We then show that a function completely monotonic (A) on each of a set of overlapping cones may be represented by a single integral, which may then be used to extend the function to the convex closure of the set of cones. Lastly, we show by an example that a function may be completely monotonic along every line with nonnegative slope in the first quadrant without being completely monotonic as a function of two variables.

2. Functions completely monotonic on cones. We commence with some notations and definitions. We shall write x in place of $(x_1, \cdots, x_n)$, xt' in place of $(x_1 t_1 + \cdots + x_n t_n)$, and where these appear in integrands we shall use a single integral sign to denote a multiple integral.

For a given convex cone D, D^* will be the set of all t such that $\sum_{i=1}^{n} x_i t_i \geq 0$ for all x in D. By an n-cone we shall mean a convex cone in E_n spanned by n linearly independent vectors $x^i = (x_1^i, \cdots, x_n^i)$, and such that there is a hyperplane having only the origin in common with

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Received August 19, 1954, and in revised form June 24, 1955. Part of this paper is a part of a dissertation submitted to Princeton University. We wish to acknowledge our gratitude to Professor S. Bochner under whose direction the dissertation was prepared. We are also grateful to the referee for several valuable suggestions.