A GEOMETRIC PROBLEM OF SHERMAN STEIN

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1. Introduction. Recently, Sherman Stein [1] has proposed the following problem:

Let $J \subset R_2$ be a rectifiable Jordan curve, with the property that for each rotation R. there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nonzero length. Must J contain the arc of a circle?

We interpret "length" to be the measure induced on J by arc length, and in § 2 we give an example to show that J need not contain the arc of a circle. In § 3 we show that if "nonzero length" is replaced by "nondegenerate component", then J must necessarily contain an arc of a circle.

2. An example. Let C be a circle in R_2, and let L be the circumference of C. Using standard arguments, we can obtain a subset D of C which is open relative to C, which is dense in C, and which has length less than $L/3$. We define J to be the point set which is obtained if we modify C by replacing each component K of D by the line segment whose end points are the end points of K. J is obviously a rectifiable Jordan curve. If R is a rotation, we choose T in such a way that TR maps C onto C. It follows that $(TRJ) \cap J$ contains $C-(D \cup TRD)$. Since D and TRD each have length less than $L/3$, we see that $(TRJ) \cap J$ has length greater than $L/3$. The curve J which we have defined satisfies the conditions of Stein's problem, but J does not contain an arc of a circle.

3. A theorem about Jordan curves. Before stating our theorem, it is convenient to prove first a key lemma about arcs in R_2. It seems to the author that this lemma is quite interesting in itself.

LEMMA. *If A and B are topological arcs in R_2 and A contains an infinite number of subarcs, each of which is congruent to B, then B is either an arc of a circle or a segment of a straight line.*

Proof. We assign natural linear orderings to A and B, and define G to be the set of all isometries of R_2 onto R_2 which map B into A. Either an infinite number of members of G are order preserving or an infinite number of members of G are order reversing, and we may

Received October 31, 1955. The results contained in this paper were obtained during a period when the author was partially supported by National Science Foundation grant NSF G 1353.