FAMILIES OF TRANSFORMATIONS IN THE FUNCTION SPACES H^p

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I. Introduction

Let the interior of the unit circle be denoted by Δ; and let the set of functions single-valued and analytic in Δ be denoted by $\mathfrak{A}$.

It is well known that certain subsets of $\mathfrak{A}$ can be made into Banach spaces by the introduction of suitable norms. In particular, if $f \in \mathfrak{A}$, and if, for $1 \leqq p \leqq \infty$,

$$\text{(I.1)} \qquad \mathscr{M}_p(f, r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p}, \qquad p < \infty$$

$$\mathscr{M}_p(f; r) = \sup_{|z|<r} |f(z)|, \qquad p = \infty$$

and if $\sup\limits_{r<1} \mathscr{M}_p(f; r) < \infty$, then f is said to be in the set H^p. Also, H^p is a Banach space with

$$\text{(I.2)} \qquad \|f\|_{H^p} = \sup_{r<1} \mathscr{M}_p(f; r)$$

A proof of these statements, together with a discussion of many properties of the spaces H^p, can be found in [8].

This paper is concerned with certain transformations in the spaces H^p[1].

Let $\omega(z)$ be a function of z which is analytic in Δ and such that $|\omega(z)| < 1$ for $z \in \Delta$. If $f \in \mathfrak{A}$, then so is the function defined by $f[\omega(z)]$. For $f \in \mathfrak{A}$, we define

$$\text{(I.3)} \qquad T_\omega f = g \underset{df}{\Longleftrightarrow} f[\omega(z)] = g(z) \text{ for } z \in \Delta.$$

T_ω is clearly an additive, homogeneous transformation.

It is well known [4] that if $f \in H^p$ and $\omega(0) = 0$, then $T_\omega f \in H^p$ and $\|T_\omega f\| \leqq \|f\|$. In other words, if $\omega(0) = 0$, then $T_\omega \in [H^p]$ (the set of all linear bounded transformations on H^p to H^p), and $\|T_\omega\| \leqq 1$. Our first task is to prove the following.

Received September 21, 1955, and revised form April 27, 1956. This paper forms a portion of Ph. D. thesis submitted at the University of California, Los Angeles. The author wishes to express his gratitude to Professor A. E. Taylor, under whose guidance the thesis was written.

[1] In the following, all statements about H^p refer to $1 \leqq p \leqq \infty$ unless further qualified.