CONTINUOUS SPECTRA AND UNITARY EQUIVALENCE

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1. Introduction. In the differential equation

$$(1) \qquad (px')'+(\lambda+f(t))x=0\ ,$$

let λ denote a real parameter and let $p(t)$ (>0) and $f(t)$ be continuous real-valued functions on $0\leqq t<\infty$. Suppose that (1) is of the limit-point type, so that (1) and a linear homogeneous boundary condition

$$(2_\alpha) \qquad x(0)\cos\alpha+x'(0)\sin\alpha=0\ , \qquad 0\leqq\alpha<\pi\ ,$$

determine a boundary value problem with a spectrum $S=S_\alpha$ on the half-line $0\leqq t<\infty$; cf. [7]. The continuous spectrum C_α (if it exists) is determined by a continuous monotone nondecreasing basis function $\rho_\alpha(\lambda)$. It is known that the set of cluster points, S', of S_α is independent of α, [7, p. 251]; the question as to whether the corresponding assertion for C_α is also true was raised by Weyl [7, 7. 252] but is still undecided.

Consider the self-adjoint operators $H_\alpha=\int\lambda dE_\alpha(\lambda)$ (all of which are extensions of the same symmetric operator) belonging to the various boundary value problems determined by (1) and (2_α); cf. for example, [2]. The object of this note is to shown that any two H_α possessing purely continuous (hence, in view of the above remark concerning S', necessarily identical) spectra are unitarily equivalent, at least if certain conditions concerning the nature of the sets C_α and the basis functions $\rho_\alpha(\lambda)$ are met. In fact there will be proved the following.

THEOREM (*). *Suppose that there exist two (distinct) values α_1 and α_2 $(0\leqq\alpha_k<\pi)$ such that, for each of the two boundary value problems determined by* (1) *and* $(2_{\alpha k})$, *the following three conditions are satisfied:*

(i) $S_{\alpha k}\neq(-\infty,\ \infty)$,

(ii) *the point spectrum is empty, and*

(iii) $\rho_{\alpha k}(\lambda)$ *is absolutely continuous. Then* $H_{\alpha 1}$ *and* $H_{\alpha 2}$ *are unitarily equivalent.*

The condition (i) of (*) surely holds if, for instance, f is bounded or even bounded from below on $0\leqq t<\infty$. It should be noted however that every (real) λ belongs to an S_α for some α (depending on λ); [1].

For other results on the continuous spectra of boundory value pro-

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