ON THE MEASURE OF NORMAL FORMULAS

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1. Introduction. Quine has recently found (in [1], [2] and [3]) a reasonably practical method which yields the simplest normal equivalent of a given truth functional formula. The problem of this paper is to find a practical method which yields the simplest normal formula with a given measure. Roughly, the measure of a formula is the number of T's in the column under the formula in a truth table which has 2^d rows; these rows represent all possible assignments of T's and F's to d letters including all the letters of the formula and perhaps others. The problem, which is rather difficult, arises in the design of certain networks in digital computers (described at the end of § 2) as part of a more general problem which is all the more difficult. Networks, however, are not discussed at all in the remainder of the paper, where the main problem is attacked as a problem in pure logic. I have had no success in obtaining a method which is generally satisfactory, but have succeeded in proving a few theorems which will probably be indispensable in any future attack on the problem.

2. The problem and its origin. Most of the terminology which I shall use is Quine's. Where it conflicts with Quine's terminology of [1], [2] and [3] I shall explicitly say so; on the other hand, I shall not presuppose that the reader is familiar with any of these papers. An italicized word appearing in a sentence of this paper is defined in that sentence. In this section a sentence without an italicized word is often a theorem which is either well known or obvious.

A formula is made up in the usual manner from the letters $A_1, \cdots, A_d$ by means of negation, conjunction and disjunction (or alternation). For any formulas $\Phi_1, \cdots, \Phi_n$, $n \geqq 2$, $\overline{\Phi}_1$ is the *negation* of Φ_1, $\Phi_1\Phi_2\cdots\Phi_n$ is the *conjunction* of $\Phi_1, \cdots, \Phi_n$ (these being *conjuncts*), and $\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_n$ is the *disjunction* (called 'alternation' by Quine) of $\Phi_1, \cdots, \Phi_n$ (these being *disjuncts*). (I assume that the reader is familiar enough with the general literature to see how the circularity of definition in the last two sentences can be avoided.) A letter or its negation is a *literal*. If a formula is a disjunction, then the disjuncts are *clauses*; if it is not a disjunction, the formula itself is its only *clause*. A formula all of whose clauses are literals or conjunctions of literals is a *normal formula*. (For Quine a clause of a normal formula cannot have

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