A CONGRUENCE THEOREM FOR TREES

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Let A and B be two trees with vertex sets $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ respectively. The trees are congruent, are isomorphic, or "are the same type", $(A \cong B)$, if there exists a one-to-one correspondence between their vertices which preserves the join-relationship between pairs of vertices. Let $c(a_i)$ denote the $(n-1)$-point subgraph of A obtained by deleting a_i and all joins (arcs, segments) at a_i from A. It is the purpose here to show that if there is a one-to-one correspondence in type, and frequency of type, between the sub-graphs of order $n-1$ in A and B, that is, if there exists a labeling such that $c(a_i) \cong c(b_i)$, $i=1, 2, \cdots, n$, then $A \cong B$. It is assumed throughout, therefore, that there is a labeling of the two trees A and B such that $c(a_i) \cong c(b_i)$, $i=1, 2, \cdots, n$, where $n \geqq 3$.

Some lemmas to the main theorem are established first. Let T denote a certain type of graph of order j, where $2 \leqq j < n$, which occurs as a subgraph α times in A and β times in B. If α_i is the number of T-type subgraphs which have a_i as a vertex, then,

$$\alpha = \left(\sum_1^n \alpha_i\right) \Big/ j.$$

Similarly,

$$\beta = \left(\sum_1^n \beta_i\right) \Big/ j,$$

where b_i is the number of T-type subgraphs having b_i as a vertex. Because $c(a_i) \cong c(b_i)$, the number of T-type subgraphs which do not have a_i as a vertex is the same as the number which do not have b_i as a vertex. Thus

$$\alpha - \alpha_i = \beta - \beta_i, \qquad i=1, 2, \cdots, n.$$

Therefore

$$\sum_1^n (\alpha - \beta) = \sum_1^n (\alpha_i - \beta_i),$$

so $n(\alpha-\beta) = j(\alpha-\beta)$, which implies $\alpha=\beta$. This, in turn, implies $\alpha_i = \beta_i$, $i=1, 2, \cdots, n$, and the lemma is established.

LEMMA 1. *Every type of proper subgraph which occurs in A or B*

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