SOME INEQUALITIES BETWEEN LATENT ROOTS AND MINIMAX (MAXIMIN) ELEMENTS OF REAL MATRICES

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1. Introduction. Because of the usual tediousness of computing latent roots, any quick information about them is often welcome and useful. We develop here some lower bounds to the absolute value of the major latent root (the one largest in absolute value) of any real symmetric matrix that depend only on a simple inspection of its elements. Also, lower bounds are developed for the largest latent root of a Gramian matrix of the form AA' that require inspection only of the elements of A. The latter case is especially important in linear regression theory of statistics, in factor analysis theories of psychology, and elsewhere.

The original motivation for our inequalities was to study the relationship between latent roots and the von Neumann value of a two-person zero-sum game matrix. We actually use the von Neumann theory to establish our bounds to latent roots, and in return we show how latent roots can be used to bound the game value of a matrix. The latter kind of bound will be useful whenever it is easier to get at the appropriate latent root than at the desired game value.

The bounds to latent roots are first exhibited in §§ 2–3, and then proved in § 4. How to reverse their emphasis to provide bounds for game values is shown in § 5.

2. Lower bounds to the major latent root. Let A be any real matrix of order $m \times n$. Let a_{ij} be the typical element of A ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$), and let p_i and q_j be defined respectively as

$$p_i = \min_j a_{ij}\,, \qquad q_j = \max_i a_{ij} \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}. \tag{1}$$

Furthermore, let p and q be defined respectively as

$$p = \max_i p_i\,, \qquad q = \min_j q_j\,. \tag{2}$$

From (1), it immediately follows that

$$p_i \leqq a_{ij} \leqq q_j \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}, \tag{3}$$

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