A REAL INVERSION FORMULA FOR A CLASS OF BILATERAL LAPLACE TRANSFORMS

WILLIAM R. GAFFEY

1. Introduction. The Post-Widder inversion formula for unilateral Laplace transformations [1] states that, under certain weak restrictions on $\phi(u)$,

$$\lim_{k\to\infty}\left(\frac{k}{c}\right)^{k+1}\frac{1}{k!}\int_0^\infty \phi(u)u^k \exp\left(-k\frac{u}{c}\right)du=\phi(c)\ ,$$

for any continuity point c of $\phi(u)$.

This formula applies when $\phi(u)$ is defined only for $u\geqq 0$. A similar formula may be deduced if $\phi(u)$ is defined for $u\geqq -a$, for some positive a. In such a case, we may let $\phi^*(u)=\phi(u-a)$, and we may then use the Post-Widder formula to determine $\phi^*(u)$ at the point $u=c+a$. The inversion formula then becomes

$$\lim_{k\to\infty}\left(\frac{k}{c+a}\right)^{k+1}\frac{1}{k!}\int_0^\infty \phi(u-a)u^k \exp\left(-k\frac{u}{c+a}\right)du=\phi(c)\ ,$$

or, if we make the transformation $z=u/(c+a)$,

$$(1)\qquad \lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_0^\infty \phi[(c+a)z-a]z^k \exp(-kz)dz=\phi(c)\ .$$

This suggests that, if $\phi(u)$ is defined for $-\infty<u<\infty$, some sort of limiting form of (1) applies. We shall prove that under suitable restrictions on ε and on the behavior of $\phi(u)$,

$$(2)\qquad \lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_{-\infty}^\infty \phi[(c+k^\varepsilon)z-k^\varepsilon]z^k \exp(-kz)dz=\phi(c)\ .$$

2. Remarks. In the following sections $\phi(u)$ will be assumed to be integrable over the interval from $-\infty$ to ∞, and c will be assumed to be a continuity point of $\phi(u)$. All limits should be understood to be for increasing values of k.

The expression $\delta/(c+k^\varepsilon)$, where δ and ε are positive numbers, occurs frequently. It will be denoted by $\delta(k, \varepsilon)$.

Finally, it may be noted that in terms of the Laplace transform of $\phi(u)$ for real t,

Received December 7, 1955, and in revised form April 13, 1956.