UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

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1. Introduction. Let D be a simply connected region with an analytic boundary C. Assume that $z=0$ is an interior point while $z=1$ lies on the boundary. We assume further that the tangent to C at $z=1$ is not parallel to the real axis. In this case, we shall be able to fit into D small angles Γ placed symmetrically about the real axis and with vertex at $z=1$. These angles will be of the form $-\delta \leqq \theta \leqq \delta$ or $\pi-\delta \leqq \theta \leqq \pi+\delta$, $\delta > 0$, depending upon the location of $z=1$. For a given $f(z)$ regular in D, we consider the following limits defined recursively

$$
\begin{aligned}
a_0 &= \lim_{z\to 1} f(z) \\
a_1 &= \lim_{z\to 1} (z-1)^{-1}[f(z)-a_0] \\
a_2 &= \lim_{z\to 1} (z-1)^{-2}[f(z)-a_0-a_1(z-1)] \\
& \cdots
\end{aligned} \tag{1}
$$

If each limit in (1) exists and is independent of the manner in which $z \to 1$ through values in some angle Γ, then $f(z)$ is said to possess an asymptotic expansion at $z=1$ in the sense of Poincaré, and this is indicated by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n (z-1)^n . \tag{2}$$

We shall designate by $A(=A(D))$ the linear class of functions which are regular in D and which possess asymptotic expansions at $z=1$ in the sense of Poincaré. The angle Γ in which (1) is valid may depend upon the particular $f \in A$ selected.

Uniqueness theory is concerned with distinguishing nontrivial subclasses of A within which the expansion $\sum_{n=0}^{\infty} a_n(z-1)^n$ determines the corresponding function uniquely. Write for the remainder

$$R_n(z)=f(z)-a_0-a_1(z-1)-\cdots-a_{n-1}(z-1)^{n-1} , \tag{3}$$

and consider the ratios

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